Physics/Astronomy 226, Problem set 1, Due 1/15
Reading: Carroll, Ch. 1

1. Consider a Euclidean space with Cartesian coordinates x^i, i.e. distances are given by $(\Delta s)^2 = \delta_{ij}\Delta x^i \Delta x^j$.

   (a) By Taylor expanding $\Delta x^{i'}(\Delta x^i)$, argue that the same formula will hold when $x^i \to x^{i'}$ if and only if

   $$x^{i'} = A^{i'}{}_i x^i + B^{i'}, \text{ where } \delta_{i'j'} A^{i'}{}_i A^{j'}{}_j = \delta_{ij}, \text{ and } A^{i'}{}_i, B^{i'} \text{ are constant.}$$

   (b) An annoying friend argues to you: "The statement that space is Euclidean is empty: given some distances, you will always be able to find *some* set of coordinates x^i such that $(\Delta s)^2 = \delta_{ij}\Delta x^i \Delta x^j$. Therefore any space could be called Euclidean." How do you prove your friend wrong? (Hint: consider N points with coordinates x^i_n, and the distances between them.)

2. The discussion of particle dynamics in class was a bit abstract, so let's do things a bit more concretely. Imagine that you are in a spacecraft traveling in one particular direction, call it the x-direction, and that you have an accelerometer on board. Imagine some inertial frame (x, t) in which you are moving. At any time, you can also set up an instantaneous rest frame (IRF) with coordinates (x', t'), in which your acceleration $d^2x'/dt'^2 = dv'/dt'$ is given by the reading $F(\tau)$ on your accelerometer, where τ is your proper time and $v = dx/dt$. (Note: the IRF is defined as an inertial frame defined so that at the relevant instant, the rocket is at rest in it, rather than a frame glued to the rocket so that the rocket is always at rest in it.)

   (a) Derive or write down the equations connecting u' to u and du'/dt' to du/dt, if $u = dx/dt$ is the velocity of some object, not necessarily the rocket (i.e. u is not necessarily equal to v.)

   (b) Find dv/dt and $dv/d\tau$ in terms of v and $F(\tau)$.

   (c) Integrate this to find $v(\tau) = \tanh\psi(\tau)$. What is $\psi(\tau)$?

   (d) Write down expressions for $dt/d\tau$ and $dx/d\tau$ in terms of $\psi(\tau)$.

   (e) To look at this another way, write down the $x-$ and $t-$ components of the 4-velocity $f^{\mu'}$ in the rocket frame. Now transform these into the unprimed frame to get an expression for $\frac{d^2t}{d\tau^2}$ and $\frac{d^2x}{d\tau^2}$ in terms of F. Confirm that your solution solves these equations.

   (f) Suppose F =const., and that at time t_0 your rocket is at position x_0. What are $x(\tau)$ and $t(\tau)$? Draw a spacetime diagram of your trajectory.

3. To continue with rocket science (which is, after all, easy compared to GR), a rocket is flying at a 3-velocity $\vec{v}_1$ in inertial frame 1. Let U^μ_1 be the spacecraft's 4-velocity in that frame. Frame 1 is moving at velocity $\vec{v}_{12}$ with respect to frame 2, with $\vec{v}_{12}$ and $\vec{v}_1$ in the same direction.

   (a) Express U^0_1 in terms of $|\vec{v}_1|$, and U^i_1 in terms of $\vec{v}_1$.

(b) Write down the Lorentz transform between frame 1 and 2 both i) in terms of $\vec{v}_{12}$ and ii) in terms of the "rapidity parameter" ϕ, where $|\vec{v}_{12}| = \tanh\phi$.

(c) Find the rocket 4-velocity U_2^μ in frame 2, and use this to deduce the standard expression for the addition of velocities (i.e. find the 3-velocity of the rocket in frame 2). Write an analogous expression in terms of ϕ.

(d) Set $|\vec{v}_1| = |\vec{v}_{12}| \equiv v$. Let frame 2 move at velocity v with respect to frame 3 (in the same direction as $\vec{v}_{12}$). Let frame 3 move at velocity v with respect to frame 4 (again in the same direction), etc. What is the 3-velocity of the rocket in frame N? (Hint: write the Lorentz transform from frame 1 to frame 3 in terms of ϕ and see what happens.).

4. Let T be a tensor with components $T^\mu{}_\nu$, and V be a vector with components V^μ.

   (a) Using the transformation rules for the tensor and vector components, prove that the components of $T^\mu{}_\nu V^\nu$ transform as a vector.

   (b) Write T and V in terms of components multiplying basis vectors $\hat{e}_{(\mu)}$ and one-forms $\hat{\theta}^{(\mu)}$. Show that $T(V)$ is a map from one-forms to $\Re$, i.e. a vector.

5. The Λ-particle is a neutral baryon of mass $M = 1115\,\mathrm{MeV}$ which decays with a lifetime of $\tau = 3 \times 10^{-10}\,\mathrm{s}$ into a nucleon of mass m_1 and a π-meson of mass m_2.

   It was first observed in flight by its charged decay mode $\Lambda \to p + \pi^-$ in cloud chambers. The two charged tracks originate from a single points. The nucleon and pion identities and momenta can be inferred from their ranges and curvature in the magnetic field of the chamber.

   (a) A Λ-particle is created with total energy 10 GeV in a collision in the top plate of a cloud chamber. How far will it on average travel in the chamber before decaying?

   (b) Derive a formula for the mass M of a decaying particle in terms of the masses m_1 and m_2 and momenta $p_1 \equiv |\vec{p}_2|$ and $p_2 \equiv |\vec{p}_2|$ of the decay products and the angle θ between the tracks in the laboratory frame.

6. Take a tensor $X^{\mu\nu}$ and vector V^μ with components (ν labels columns and μ labels rows):

$$X^{\mu\nu} = \begin{pmatrix} 1 & 2 & -2 & 1 \\ -2 & 0 & 2 & 2 \\ -1 & 1 & 1 & 0 \\ -2 & 0 & 1 & 1 \end{pmatrix}, \quad V^\mu = (1, -2, 0, 1)\,.$$

   Let the metric be $\eta_{\mu\nu}$, and consider each of the following. For each valid tensor equation, evaluate the l.h.s. For each invalid equation, state why it is invalid.

   (a) $Y = X^\mu{}_\mu$

   (b) $Z = X_{\mu\mu}$

   (c) $V = V^\mu V_\mu$

(d) $T^{\nu} = X^{\mu\nu}V_{\mu}$

(e) $Q^{\mu\nu} = X^{\mu}{}_{\alpha}X^{\alpha\nu} + X^{(\mu\nu)}$

(f) $G_{\mu\nu\alpha\beta} = X_{\mu\nu} + V_{\alpha}X_{\beta\delta}V^{\delta}$

(g) $R^{[\mu\nu]} = X^{[\mu\nu]} - V^{[\mu}V^{\nu]}$