Physics/Astronomy 226, Problem set 1, Due 1/15
Reading: Carroll, Ch. 1
Solutions

1. Consider a Euclidean space with Cartesian coordinates x^i, i.e. distances are given by $(\Delta s)^2 = \delta_{ij}\Delta x^i \Delta x^j$.

   (a) By Taylor expanding $\Delta x^{i'}(\Delta x^i)$, argue that the same formula will hold when $x^i \to x^{i'}$ if and only if

   $$x^{i'} = A^{i'}{}_i x^i + B^{i'}, \text{ where } \delta_{i'j'}A^{i'}{}_i A^{j'}{}_j = \delta_{ij}, \text{ and } A^{i'}{}_i, B^{i'} \text{ are constant.}$$

   (b) An annoying friend argues to you: "The statement that space is Euclidean is empty: given some distances, you will always be able to find *some* set of coordinates x^i such that $(\Delta s)^2 = \delta_{ij}\Delta x^i \Delta x^j$. Therefore any space could be called Euclidean." How do you prove your friend wrong? (Hint: consider N points with coordinates x^i_n, and the distances between them.)

Solution:

(a) This problem is a mess to write out but good practice in manipulating little indices. One direction of the "if and only if" was essentially given in class: *If*

$$x^{i'} = A^{i'}{}_i x^i + B^{i'}, \text{ where } \delta_{i'j'}A^{i'}{}_i A^{j'}{}_j = \delta_{ij}, \text{ and } A^{i'}{}_i, B^{i'} \text{ are constant,}$$

then

$$\Delta x^{i'} = A^{i'}{}_i \Delta x^i.$$

Calculating $\delta_{i'j'}\Delta x^{i'}\Delta x^{j'}$, we find

$$\delta_{i'j'}\Delta x^{i'}\Delta x^{j'} = \delta_{i'j'}A^{i'}{}_i A^{j'}{}_j \Delta x^i \Delta x^j = \delta_{ij}\Delta x^i \Delta x^j.$$

Now, for the other direction, first we can consider $\Delta x^{i'}$ to be a function of Δx^i and Taylor expand around $\Delta x^i = 0$:

$$\Delta x^{i'} = \frac{\partial x^{i'}}{\partial x^i}\Delta x^i + \frac{1}{2}\frac{\partial^2 x^{i'}}{\partial x^j \partial x^k}\Delta x^j \Delta x^k + ...$$

Now if we assume that $(\Delta s)^2 = \delta_{ij}\Delta x^i \Delta x^j = \delta_{i'j'}\Delta x^{i'}\Delta x^{j'}$, we can plug in the Taylor expansion to get the somewhat ghastly expression:

$$\begin{aligned}
\delta_{i'j'}\Delta x^{i'}\Delta x^{j'} &= \delta_{i'j'}\left[\frac{\partial x^{i'}}{\partial x^i}\Delta x^i + \frac{1}{2}\frac{\partial^2 x^{i'}}{\partial x^k \partial x^l}\Delta x^k \Delta x^l + ...\right] \\
&\times \left[\frac{\partial x^{j'}}{\partial x^j}\Delta x^j + \frac{1}{2}\frac{\partial^2 x^{j'}}{\partial x^m \partial x^n}\Delta x^m \Delta x^n + ...\right] \\
&= \delta_{i'j'}\frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^{j'}}{\partial x^j}\Delta x^i \Delta x^j + \frac{1}{2}\delta_{i'j'}\frac{\partial x^{i'}}{\partial x^i}\frac{\partial^2 x^{j'}}{\partial x^m \partial x^n}\Delta x^i \Delta x^m \Delta x^n + ...
\end{aligned}$$

This can only work for a general set of Δx^ms if the second derivative terms such as $\partial^2 x^{j'}/\partial x^m \partial x^n$ are zero. But this means that $\partial x^{i'}/\partial x^i$ are constant, so we can substitute the constant matrices $A^{i'}{}_i$ as above. By setting the terms in $\Delta x^i \Delta x^j$ equal we recover the orthogonality condition, and of course we can add a constant $B^{i'}$ to each term without affecting the $\Delta x^{i'}$ at all. Thus the "only if" is proved.

(b) Let's look at three dimensions. Consider N points with coordinates x^i_n, $n = 1..N$. We can give each point any coordinates we like, so there are $3N$ numbers we may freely specify. But we can express the distance between any two points as $(\Delta s_{ab})^2 = \delta_{ij}(x^i_a - x^i_b)(x^j_a - x^j_b)$, where a and b label any two points. These constitute $N(N-1)/2$ equations. If we consider the distances fixed, we have only $3N$ variables to play with by changing variables, so we will not be able to find a solution if (you can readily show) $N > 7$.

2. The discussion of particle dynamics in class was a bit abstract, so let's do things a bit more concretely. Imagine that you are in a spacecraft traveling in one particular direction, call it the x-direction, and that you have an accelerometer on board. Imagine some inertial frame (x, t) in which you are moving. At any time, you can also set up an instantaneous rest frame (IRF) with coordinates (x', t'), in which your acceleration $d^2x'/dt'^2 = dv'/dt'$ is given by the reading $F(\tau)$ on your accelerometer, where τ is your proper time and $v = dx/dt$. (Note: the IRF is defined as an inertial frame defined so that at the relevant instant, the rocket is at rest in it, rather than a frame glued to the rocket so that the rocket is always at rest in it.)

   (a) Derive or write down the equations connecting u' to u and du'/dt' to du/dt, if $u = dx/dt$ is the velocity of some object, not necessarily the rocket (i.e. u is not necessarily equal to v.)

   (b) Find dv/dt and $dv/d\tau$ in terms of v and $F(\tau)$.

   (c) Integrate this to find $v(\tau) = \tanh \psi(\tau)$. What is $\psi(\tau)$?

   (d) Write down expressions for $dt/d\tau$ and $dx/d\tau$ in terms of $\psi(\tau)$.

   (e) To look at this another way, write down the $x-$ and $t-$ components of the 4-velocity $f^{\mu'}$ in the rocket frame. Now transform these into the unprimed frame to get an expression for $\frac{d^2t}{d\tau^2}$ and $\frac{d^2x}{d\tau^2}$ in terms of F. Confirm that your solution solves these equations.

   (f) Suppose F =const., and that at time t_0 your rocket is at position x_0. What are $x(\tau)$ and $t(\tau)$? Draw a spacetime diagram of your trajectory.

Solution:

(a) Let the rocket be moving in the $+x$ direction, so $v = dx/dt$. From the usual Lorentz transform, $t' = \gamma(t - vx)$ and $x' = \gamma(x - vt)$, so

$$dt' = \gamma(dt - vdx) = \gamma(1 - uv)dt, \quad dx' = \gamma(dx - vdt).$$

Therefore

$$u' = \frac{dx'}{dt'} = \frac{u - v}{1 - uv}.$$

Taking another differential,

$$du' = \frac{du}{\gamma^2(1 - uv)^2},$$

so

$$\frac{du'}{dt'} = \frac{du/dt}{\gamma^3(1-uv)^3}.$$

(b) Now we set $u = v$, so in the IRF (where $dt' = d\tau$), we have

$$\frac{dv'}{dt'} = \frac{dv'}{d\tau'} = F(\tau).$$

Using our formulas from part (a),

$$\frac{dv}{dt} = (1-v^2)^{3/2}F(\tau),$$

and

$$\frac{du}{d\tau} = (1-u^2)F(\tau).$$

(c) We find

$$v(\tau) = \tanh\psi(\tau),$$

where $\psi(\tau) = \int_0^\tau F(\tau')d\tau'$ if $v(\tau = 0) = 0$.

(d) Sticking these in to the formula for γ we find:

$$dt/d\tau = \gamma = \cosh\psi(\tau)$$

and similarly

$$dx/d\tau = v\gamma = \sinh\psi(\tau).$$

(e) In the primed frame, we have as per the class discussion $f^{0'} = 0$, $f^{x'} = F(\tau)$. Then

$$\frac{d^2t}{d\tau^2} = f^0 = \Lambda^0{}_{0'}f^{0'} + \Lambda^0{}_{1'}f^{1'} = v\gamma F(\tau),$$
$$\frac{d^2x}{d\tau^2} = f^1 = \Lambda^1{}_{0'}f^{0'} + \Lambda^1{}_{1'}f^{1'} = \gamma F(\tau).$$

If we take a τ derivative of our result of part (d), we get

$$d^2t/d\tau^2 = \frac{d}{d\tau}\cosh\psi = (\sinh\psi)F = \gamma v F,$$

and

$$d^2x/d\tau^2 = \frac{d}{d\tau}\cosh\psi = (\cosh\psi)F = \gamma F,$$

so it all checks out.

(f) The equations integrate to:

$$t = t_0 + \frac{1}{F}\sinh F\tau,$$

$$x = x_0 + \frac{1}{F}(\cosh f\tau - 1)$$

This looks like a timelike hyperbola that asymptotes to the line $t - t_0 = x - x_0$.

3. To continue with rocket science (which is, after all, easy compared to GR), a rocket is flying at a 3-velocity $\vec{v}_1$ in inertial frame 1. Let U_1^μ be the spacecraft's 4-velocity in that frame. Frame 1 is moving at velocity $\vec{v}_{12}$ with respect to frame 2, with $\vec{v}_{12}$ and $\vec{v}_1$ in the same direction.

   (a) Express U_1^0 in terms of $|\vec{v}_1|$, and U_1^i in terms of $\vec{v}_1$.

   (b) Write down the Lorentz transform between frame 1 and 2 both i) in terms of $\vec{v}_{12}$ and ii) in terms of the "rapidity parameter" ϕ, where $|\vec{v}_{12}| = \tanh\phi$.

   (c) Find the rocket 4-velocity U_2^μ in frame 2, and use this to deduce the standard expression for the addition of velocities (i.e. find the 3-velocity of the rocket in frame 2). Write an analogous expression in terms of ϕ.

   (d) Set $|\vec{v}_1| = |\vec{v}_{12}| \equiv v$. Let frame 2 move at velocity v with respect to frame 3 (in the same direction as $\vec{v}_{12}$). Let frame 3 move at velocity v with respect to frame 4 (again in the same direction), etc. What is the 3-velocity of the rocket in frame N? (Hint: write the Lorentz transform from frame 1 to frame 3 in terms of ϕ and see what happens.).

Solution:

(a) Beginning with the equation of relativistic time dilation:

$$t = \gamma\tau = \frac{\tau}{\sqrt{1-v_1^2}}$$

we find that ...
$$U_1^0 = \frac{\partial t}{\partial\tau} = \gamma = \frac{1}{\sqrt{1-v_1^2}}$$

And from the equation of length contraction:

$$\begin{aligned} x_1^i &= x^i\gamma^{-1} = x^i\sqrt{1-v_1^2} \\ v_1^i &= \frac{\partial x^i}{\partial t} = \frac{\partial x^i}{\partial\tau}\frac{\partial\tau}{\partial t} = \frac{U_1^i}{U_1^0} \\ \Rightarrow U_1^i &= v_1^i U_1^0 = \frac{v_1}{\sqrt{1-v_1^2}} \end{aligned}$$

Note: this can also be done by knowing that the rocket has 4-velocity $U_0^\mu = (1,0,0,0)$ in its rest frame, then applying to this a boost of velocity $\vec{v}_1$.

(b) First, chose the direction of motion of the rocket to be in the x-direction. Then, beginning from the equations of length contraction and time dilation:

$$\begin{aligned} x^{i'} &= (x^i + v\,\tau)\,\gamma \\ t &= (\tau + x^i\,v)\,\gamma \end{aligned}$$

We may write the Lorentz transformation as ...

$$\Rightarrow \Lambda^1{}_2 = \begin{pmatrix} \gamma_{12} & v_{12}\,\gamma_{12} & 0 & 0 \\ v_{12}\,\gamma_{12} & \gamma_{12} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\gamma_{12} = (1 - |\vec{v}_{12}|^2)^{-1/2}$.
If $|\vec{v}_{12}| = \tanh\phi$ then ...
$\gamma_{12} = (1 - \tanh^2\phi)^{-\frac{1}{2}} = \cosh\phi$
and since $|\vec{v}_{12}|\,\gamma = \tanh\phi \cdot \cosh\phi = \sinh\phi$

$$\Rightarrow \Lambda^1{}_2 = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(c) Let $v_{12} = |\vec{v}_{12}|$. Then

$$U^\mu{}_2 = \Lambda^1{}_2\, U^\mu{}_1 = \begin{pmatrix} \gamma_{12} & v_{12}\,\gamma_{12} & 0 & 0 \\ v_{12}\,\gamma_{12} & \gamma_{12} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ v_1\,\gamma_1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_1\,\gamma_{12}(1 + v_1\,v_{12}) \\ \gamma_1\,\gamma_{12}(v_1 + v_{12}) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial t}{\partial\tau} \\ \frac{\partial x^1}{\partial\tau} \\ \frac{\partial x^2}{\partial\tau} \\ \frac{\partial x^3}{\partial\tau} \end{pmatrix}$$

so

$$v' = \frac{\partial x^1}{\partial t} = \frac{\partial x^1}{\partial\tau}\,\frac{\partial\tau}{\partial t}$$

$$\Rightarrow v' = \frac{\gamma_1\,\gamma_{12}(v_1 + v_{12})}{\gamma_1\,\gamma_{12}(1 + v_1\,v_{12})} = \frac{v_1 + v_{12}}{1 + v_1\,v_{12}}$$

or ...

$$U^\mu{}_2 = \Lambda^1{}_2\, U^\mu{}_1 = \begin{pmatrix} \cosh\phi_{12} & \sinh\phi_{12} & 0 & 0 \\ \sinh\phi_{12} & \cosh\phi_{12} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\phi_1 \\ \sinh\phi_1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh\phi_1\ \cosh\phi_{12} + \sinh\phi_1\ \sinh\phi_{12} \\ \cosh\phi_1\ \sinh\phi_{12} + \sinh\phi_1\ \cosh\phi_{12} \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow v' = \frac{\cosh\phi_1\ \sinh\phi_{12} + \sinh\phi_1\ \cosh\phi_{12}}{\cosh\phi_1\ \cosh\phi_{12} + \sinh\phi_1\ \sinh\phi_{12}} = \frac{\sinh(\phi_1 + \phi_{12})}{\cosh(\phi_1 + \phi_{12})} = \tanh(\phi_1 + \phi_{12})$$

(d) So, switching to the rapidity parameter ϕ, if $|v_1| = |v_{12}| = v$ then $\phi_1 = \phi_{12} = \phi_{23} = \phi$

thus

$$\begin{aligned}
\Lambda^1{}_3 &= \begin{pmatrix} \cosh(\phi+\phi) & \sinh(\phi+\phi) & 0 & 0 \\ \sinh(\phi+\phi) & \cosh(\phi+\phi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
\Rightarrow U^\mu{}_3 = \Lambda^1{}_3 U^\mu{}_1 &= \begin{pmatrix} \cosh 2\phi & \sinh 2\phi & 0 & 0 \\ \sinh 2\phi & \cosh 2\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\phi \\ \sinh\phi \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \cosh 2\phi \cosh\phi + \sinh 2\phi \sinh\phi \\ \sinh 2\phi \cosh\phi + \cosh 2\phi \sinh\phi \\ 0 \\ 0 \end{pmatrix} \\
\Rightarrow v_3' &= \tanh(2\phi+\phi) = \tanh(3\phi) \\
\Rightarrow v_N' &= \tanh(N\phi)
\end{aligned}$$

and since $\quad \phi = \tanh^{-1} v \ \ldots$
$v_N = \tanh(N \cdot \tanh^{-1} v)$
We can write this in terms of v by utilizing the trigonometric identity: $\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x} \qquad \text{for } x^2 < 1$
We see that

$$\begin{aligned}
v_N &= \tanh\left(ln \left(\frac{1+v}{1-v} \right)^{\frac{N}{2}} \right) \\
\Rightarrow v_N &= \frac{1 - (\frac{1-v}{1+v})^N}{1 + (\frac{1-v}{1+v})^N}.
\end{aligned}$$

Thus $v_N \to 1$ as $N \to \infty$.

4. Let T be a tensor with components $T^\mu{}_\nu$, and V be a vector with components V^μ.

   (a) Using the transformation rules for the tensor and vector components, prove that the components of $T^\mu{}_\nu V^\nu$ transform as a vector.

   (b) Write T and V in terms of components multiplying basis vectors $\hat{e}_{(\mu)}$ and one-forms $\hat{\theta}^{(\mu)}$. Show that $T(V)$ is a map from one-forms to $\Re$, i.e. a vector.

Solution: (a) We know that

$$V^{\mu'} = \Lambda^{\mu'}{}_\mu V^\mu \quad \text{and} \quad T^{\mu'}{}_{\nu'} = \Lambda^{\mu'}{}_\mu \Lambda^\nu{}_{\nu'} T^\mu{}_\nu.$$

Then

$$T^{\mu'}{}_{\nu'} V^{\nu'} = \Lambda^{\mu'}{}_\mu \Lambda^\nu{}_{\nu'} T^\mu{}_\nu \Lambda^{\nu'}{}_\alpha V^\alpha = \delta^\nu_\alpha \Lambda^{\mu'}{}_\mu T^\mu{}_\nu V^\alpha = \Lambda^{\mu'}{}_\mu T^\mu{}_\nu V^\nu,$$

which is the vector transformation law. Note the introduction of the dummy index α in the first equality, and the use of the fact that $\Lambda^\nu{}_{\nu'}$ and $\Lambda^{\nu'}{}_\alpha$ are inverse matrixes in the second equality.

(b) This time,

$$V = V^\mu \hat{e}_\mu \quad \text{and} \quad T = T^\mu{}_\nu \hat{e}_{(\mu)} \otimes \hat{\theta}^{(\nu)}.$$

Then T acting on V gives

$$\begin{aligned} & T^\mu{}_\nu \hat{e}_{(\mu)} \otimes \hat{\theta}^{(\nu)}(V^\alpha \hat{e}_{(\alpha)}) \\ = \; & T^\mu{}_\nu \hat{e}_{(\mu)} \otimes V^\alpha \hat{\theta}^{(\nu)}(\hat{e}_{(\alpha)}) \\ = \; & T^\mu{}_\nu V^\alpha \hat{e}_{(\mu)} \otimes \delta^\nu_\alpha \\ = \; & T^\mu{}_\nu V^\nu \hat{e}_{(\mu)}. \end{aligned}$$

The first line holds because only the one-forms θ will act on the vector V. The second line comes from the linearity of one forms. The third follows from the definition of the tensor product and the choice of the natural basis one-forms θ. To get the last line we can remove the tensor product because δ^ν_α is just a set of scalars, for which tensor multiplication is just multiplication. The last line is manifestly a vector.

5. The Λ-particle is a neutral baryon of mass $M = 1115\,\text{MeV}$ which decays with a lifetime of $\tau = 3 \times 10^{-10}\,\text{s}$ into a nucleon of mass m_1 and a π-meson of mass m_2.

   It was first observed in flight by its charged decay mode $\Lambda \to p + \pi^-$ in cloud chambers. The two charged tracks originate from a single points. The nucleon and pion identities and momenta can be inferred from their ranges and curvature in the magnetic field of the chamber.

   (a) A Λ-particle is created with total energy 10 GeV in a collision in the top plate of a cloud chamber. How far will it on average travel in the chamber before decaying?

   (b) Derive a formula for the mass M of a decaying particle in terms of the masses m_1 and m_2 and momenta $p_1 \equiv |\vec{p}_2|$ and $p_2 \equiv |\vec{p}_2|$ of the decay products and the angle θ between the tracks in the laboratory frame.

   Solution:

   (a) From $-m^2 = p_\mu p^\mu$ We have

   $$E^2 = p^2 + m^2$$

   where $p = |p^i| = \gamma m \vec{v}$, where v is the 3-velocity of the particle. Then, $\gamma v = [(E/m)^2 - 1]^{1/2}$, and $t = \gamma\tau$ give

   $$d = vt = [(E/m)^2 - 1]^{1/2}\tau = [(10\,GeV/1115\,MeV)^2 - 1]^{1/2} \times c \times 3 \times 10^{-10} s \simeq 80\,cm.$$

   (b) $p^\mu = p_1^\mu + p_2^\mu$, where 1,2 index the decay products. Then $m^2 = p^\mu p_\mu$ gives

   $$\begin{aligned} m^2 &= p_1^\mu p_{1,\mu} + p_2^\mu p_{2,\mu} + 2p_1^\mu p_{2,\mu} \\ &= m_1^2 + m_2^2 + 2(E_1 E_2 - \vec{p}_1 \cdot \vec{p}_2) \\ &= m_1^2 + m_2^2 + 2\left(\sqrt{(m_1^2 + p_1^2)(m_2^2 + p_2^2)} - p_1 p_2 \cos\theta\right). \end{aligned} \tag{1}$$

6. Take a tensor $X^{\mu\nu}$ and vector V^μ with components (ν labels columns and μ labels rows):

$$X^{\mu\nu} = \begin{pmatrix} 1 & 2 & -2 & 1 \\ -2 & 0 & 2 & 2 \\ -1 & 1 & 1 & 0 \\ -2 & 0 & 1 & 1 \end{pmatrix}, \quad V^\mu = (1, -2, 0, 1)\,.$$

Let the metric be $\eta_{\mu\nu}$, and consider each of the following. For each valid tensor equation, evaluate the l.h.s. For each invalid equation, state why it is invalid.

(a) $Y = X^\mu{}_\mu$

(b) $Z = X_{\mu\mu}$

(c) $V = V^\mu V_\mu$

(d) $T^\nu = X^{\mu\nu} V_\mu$

(e) $Q^{\mu\nu} = X^\mu{}_\alpha X^{\alpha\nu} + X^{(\mu\nu)}$

(f) $G_{\mu\nu\alpha\beta} = X_{\mu\nu} + V_\alpha X_{\beta\delta} V^\delta$

(g) $R^{[\mu\nu]} = X^{[\mu\nu]} - V^{[\mu} V^{\nu]}$

Solution:

(a) $Y = \eta_{\mu\nu} X^{\mu\nu}$, which is perfectly OK. To compute it, you know that you will only get a contribution to the sum then $\mu = \nu$, so you can write it as:

$$Y = \eta_{00} X^{00} + \eta_{11} X^{11} + \eta_{22} X^{22} + \eta_{33} X^{33} = -1 + 0 + 1 + 1 = 1.$$

(b) This is *not* OK as we are contracting over two lower indices. In this case the result will depend on the coordinate system (can you see why? the coordinate transformations will *not* form $\delta_{\mu\nu}$ when multiplied), whereas the l.h.s. will not.

(c) $V = \eta_{\mu\nu} V^\mu V^\nu = -1 + 4 + 0 + 1 = 4$.

(d) This is a valid equation but also a good reminder that we have to be careful in converting to matrix notation in order to do the actual computation. First, we need to lower the index of V, giving $V_u = (-1, -2, 0, 1)$. Then, we must multiply in by our matrix so that for each ν, we multiply V_u by the *column* of $X^{\mu\nu}$, i.e.

$$T^0 = -1 \times 1 + -2 \times -2 + 0 \times -1 + 1 \times -2 = 1.$$

We then end up with

$$T^\nu = (1, -2, -1, -4).$$

(e) This is also fine. If we write the first term as $X^{\mu\alpha} \eta_{\alpha\beta} X^{\beta\nu}$, it is straightforward matrix multiplication $X \eta X$ (considered as matrices). The second term is a symmetrization. We thus get:

$$Q^{\mu\nu} = \begin{pmatrix} -5 & -4 & 5 & 4 \\ -4 & 6 & 0 & 4 \\ -2 & 3 & 1 & 3 \\ -1 & 5 & -2 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 & -1.5 & -0.5 \\ 0 & 0 & 1.5 & 1 \\ -1.5 & 1.5 & 1 & 0.5 \\ -0.5 & 1 & 0.5 & 1 \end{pmatrix} = \begin{pmatrix} -4 & -4 & 3.5 & 3.5 \\ -4 & 6 & 1.5 & 5 \\ -3.5 & 4.5 & 2 & 3.5 \\ -1.5 & 6 & -1.5 & 4 \end{pmatrix}$$

(f) This is no good as it equates a (0 4) tensor to a (malformed) sum of (0 2) tensors.

(g) The second term vanishes because $V^\mu V^\nu$ is symmetric, but then we anti-symmetrize over it. So we just have to anti-symmetrize $X^{\mu\nu}$ to get:

$$R^{[\mu\nu]} = \frac{1}{2}\left[X^{\mu\nu} - X^{\nu\mu}\right] = \frac{1}{2}\begin{pmatrix} 0 & 4 & -1 & 3 \\ -4 & 0 & 1 & 2 \\ 1 & -1 & 0 & -1 \\ -3 & -2 & 1 & 0 \end{pmatrix}.$$

Note that we don't actually know what $R^{\mu\nu}$ is.