Physics/Astronomy 226, Problem set 2, Due 1/27

1. (Repost from PS1) Take a tensor $X^{\mu\nu}$ and vector V^μ with components (ν labels columns and μ labels rows):

$$X^{\mu\nu} = \begin{pmatrix} 1 & 2 & -2 & 1 \\ -2 & 0 & 2 & 2 \\ -1 & 1 & 1 & 0 \\ -2 & 0 & 1 & 1 \end{pmatrix}, \quad V^\mu = (1, -2, 0, 1)\,.$$

Let the metric be $\eta_{\mu\nu}$, and consider each of the following. For each valid tensor equation, evaluate the l.h.s. For each invalid equation, state why it is invalid.

(a) $Y = X^\mu{}_\mu$

(b) $Z = X_{\mu\mu}$

(c) $V = V^\mu V_\mu$

(d) $T^\nu = X^{\mu\nu} V_\mu$

(e) $Q^{\mu\nu} = X^\mu{}_\alpha X^{\alpha\nu} + X^{(\mu\nu)}$

(f) $G_{\mu\nu\alpha\beta} = X_{\mu\nu} + V_\alpha X_{\beta\delta} V^\delta$

(g) $R^{[\mu\nu]} = X^{[\mu\nu]} - V^{[\mu} V^{\nu]}$

2. (a) Show that the equation

$$\tilde{\epsilon}^{\beta\alpha\mu\nu} \partial_\alpha F_{\mu\nu} = 0$$

is equivalent to the Maxwell equations

$$\partial_i B^i = 0 \quad \text{and} \quad \tilde{\epsilon}^{ijk} \partial_j E_k + \partial_0 B^i = 0.$$

(b) Show that it is also equivalent to the two alternative forms

$$\partial_{[\alpha} F_{\mu\nu]} = 0 \quad \text{or} \quad \partial_\alpha F_{\mu\nu} + \partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} = 0.$$

3. Calculate the nonzero components of the energy-momentum tensor $T^{\mu\nu}$ in cartesian coordinates in an inertial frame in which there is a flat disk of radius r_0 composed of N particles of mass m, rotating counterclockwise in the $x - y$ plane about some fixed point, with fixed (radius-independent) angular velocity ω. Assume that the thickness of the disk is $\ll r_0$, and that N is large enough that one can treat the particles as continuously distributed with fixed number density in the rest frame of the disk. (Notes: (i) Don't worry about what is keeping the particles rotating like this. (ii) Nor should you worry about the effect of their mass on the spacetime – assume it is Minkowski. (iii) Also, you can express your answer using phrases like "inside the disk" and "outside the disk". (iv) Assume that the particle number density n is uniform in the rest frame of the disk. (v) Express you answer in Cartesian coordinates.)

Now suppose there is another such disk present with the same radius and center-of-rotation but with angular velocity $-\omega$, and that the particles do not collide or interact in any way. What is $T^{\mu\nu}$ in this case?