Physics/Astronomy 226, Problem set 2, Due 1/27
Solutions

1. (Repost from PS1) Take a tensor $X^{\mu\nu}$ and vector V^μ with components (ν labels columns and μ labels rows):

$$X^{\mu\nu} = \begin{pmatrix} 1 & 2 & -2 & 1 \\ -2 & 0 & 2 & 2 \\ -1 & 1 & 1 & 0 \\ -2 & 0 & 1 & 1 \end{pmatrix}, \quad V^\mu = (1, -2, 0, 1)\,.$$

Let the metric be $\eta_{\mu\nu}$, and consider each of the following. For each valid tensor equation, evaluate the l.h.s. For each invalid equation, state why it is invalid.

(a) $Y = X^\mu{}_\mu$

(b) $Z = X_{\mu\mu}$

(c) $V = V^\mu V_\mu$

(d) $T^\nu = X^{\mu\nu} V_\mu$

(e) $Q^{\mu\nu} = X^\mu{}_\alpha X^{\alpha\nu} + X^{(\mu\nu)}$

(f) $G_{\mu\nu\alpha\beta} = X_{\mu\nu} + V_\alpha X_{\beta\delta} V^\delta$

(g) $R^{[\mu\nu]} = X^{[\mu\nu]} - V^{[\mu} V^{\nu]}$

Solution:

(a) $Y = \eta_{\mu\nu} X^{\mu\nu}$, which is perfectly OK. To compute it, you know that you will only get a contribution to the sum then $\mu = \nu$, so you can write it as:

$$Y = \eta_{00} X^{00} + \eta_{11} X^{11} + \eta_{22} X^{22} + \eta_{33} X^{33} = -1 + 0 + 1 + 1 = 1.$$

(b) This is *not* OK as we are contracting over two lower indices. In this case the result will depend on the coordinate system (can you see why? the coordinate transformations will *not* form $\delta_{\mu\nu}$ when multiplied), whereas the l.h.s. will not.

(c) $V = \eta_{\mu\nu} V^\mu V^\nu = -1 + 4 + 0 + 1 = 4.$

(d) This is a valid equation but also a good reminder that we have to be careful in converting to matrix notation in order to do the actual computation. First, we need to lower the index of V, giving $V_u = (-1, -2, 0, 1)$. Then, we must multiply in by our matrix so that for each ν, we multiply V_u by the *column* of $X^{\mu\nu}$, i.e.

$$T^0 = -1 \times 1 + -2 \times -2 + 0 \times -1 + 1 \times -2 = 1.$$

We then end up with

$$T^\nu = (1, -2, -1, -4).$$

(e) This is also fine. If we write the first term as $X^{\mu\alpha} \eta_{\alpha\beta} X^{\beta\nu}$, it is straightforward matrix multiplication $X\eta X$ (considered as matrices). The second term is a symmetrization. We thus get:

$$Q^{\mu\nu} = \begin{pmatrix} -5 & -4 & 5 & 4 \\ -4 & 6 & 0 & 4 \\ -2 & 3 & 1 & 3 \\ -1 & 5 & -2 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 & -1.5 & -0.5 \\ 0 & 0 & 1.5 & 1 \\ -1.5 & 1.5 & 1 & 0.5 \\ -0.5 & 1 & 0.5 & 1 \end{pmatrix} = \begin{pmatrix} -4 & -4 & 3.5 & 3.5 \\ -4 & 6 & 1.5 & 5 \\ -3.5 & 4.5 & 2 & 3.5 \\ -1.5 & 6 & -1.5 & 4 \end{pmatrix}$$

(f) This is no good as it equates a (0 4) tensor to a (malformed) sum of (0 2) tensors.

(g) The second term vanishes because $V^\mu V^\nu$ is symmetric, but then we anti-symmetrize over it. So we just have to anti-symmetrize $X^{\mu\nu}$ to get:

$$R^{[\mu\nu]} = \frac{1}{2}\left[X^{\mu\nu} - X^{\nu\mu}\right] = \frac{1}{2}\begin{pmatrix} 0 & 4 & -1 & 3 \\ -4 & 0 & 1 & 2 \\ 1 & -1 & 0 & -1 \\ -3 & -2 & 1 & 0 \end{pmatrix}.$$

Note that we don't actually know what $R^{\mu\nu}$ is.

2. (a) Show that the equation

$$\tilde{\epsilon}^{\beta\alpha\mu\nu}\partial_\alpha F_{\mu\nu} = 0$$

is equivalent to the Maxwell equations

$$\partial_i B^i = 0 \quad \text{and} \quad \tilde{\epsilon}^{ijk}\partial_j E_k + \partial_0 B^i = 0.$$

(b) Show that it is also equivalent to the two alternative forms

$$\partial_{[\alpha}F_{\mu\nu]} = 0 \quad \text{or} \quad \partial_\alpha F_{\mu\nu} + \partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} = 0.$$

Solution:

(a) We can have either $\beta = 0$ or $\beta = i$ where $i = 1, 2, 3$. Beginning with $\beta = 0$, we notice that $\tilde{\epsilon}^{0\alpha\mu\nu} = \epsilon^{ijk}$, where i, j, k are spatial dimensions. The equation above then simplifies to

$$\tilde{\epsilon}^{ijk}\partial_i F_{jk} = \partial_i(\tilde{\epsilon}^{ijk}F_{jk}) = 2\partial_i B^i = \partial_i B^i = 0.$$

Now setting $\beta = i$, we have

$$\tilde{\epsilon}^{i\alpha\mu\nu}\partial_\alpha F_{\mu\nu} = \tilde{\epsilon}^{i0\mu\nu}\partial_0 F_{\mu\nu} - \tilde{\epsilon}^{ij\mu\nu}\partial_j F_{\mu\nu}. \tag{1}$$

On the first term on the RHS, we get contributions only from $\mu, \nu = j, k$ (i.e. spatial indices), so that

$$\tilde{\epsilon}^{i0\mu\nu}\partial_0 F_{\mu\nu} = -\tilde{\epsilon}^{0ijk}\partial_0 F_{jk} = -\partial_0(\epsilon^{ijk}F_{jk}) = -2\partial_0 B^i.$$

In the second term on the RHS of Eq. 1, we must perform the analysis with μ and ν set to zero in separate cases and add the resulting two terms, but since both the Levi-Civita symbol and field-strength tensor are anti-symmetric, we can just calculate it in one case and multiply by two to get the final answer. Proceeding with $\mu = 0$,

$$\tilde{\epsilon}^{ij\mu\nu}\partial_j F_{\mu\nu} \rightarrow \tilde{\epsilon}^{ij0k}\partial_j F_{0k} = \tilde{\epsilon}^{ijk}\partial_j F_0 k = \tilde{\epsilon}^{ijk}\partial_j E_k.$$

Thus, combining these (and remembering the our original equation equals zero so we can drop both the factors of two):

$$\tilde{\epsilon}^{i\alpha\mu\nu}\partial_\alpha F_{\mu\nu} = \partial_0 B^i + \tilde{\epsilon}^{ijk}\partial_j E_k = 0.$$

(b) First, we'll show that the two equations listed above are the same:

$$\begin{aligned}\partial_{[\alpha}F_{\mu\nu]} &= \frac{1}{6}(\partial_\alpha F_{\mu\nu} - \partial_\mu F_{\alpha\nu} + \partial_\mu F_{\nu\alpha} - \partial_\alpha F_{\nu\mu} + \partial_\nu F_{\alpha\mu} - \partial_\nu F_{\mu\alpha}) \\ &= \frac{1}{3}(\partial_\alpha F_{\mu\nu} + \partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu}) = 0.\end{aligned}$$

where we have exploited the anti-symmetric property of the field-strength tensor. Obviously the factor of $1/3$ is irrelevant since the whole thing is equal to zero. With this done, we just have to show that one of these two equations equals the equation given. This is easier to see with the more compact version. Since we began with $\tilde{\epsilon}^{\beta\alpha\mu\nu}\partial_\alpha F_{\mu\nu}$, the effect of the Levi-Civita symbol is to anti-symmetrize the indices that it contracts with. Thus, another way to write this equation showing the anti-symmetric relations of the α, μ, and ν indices without explicitly showing the LC symbol is to simply write it as $\partial_{[\alpha}F_{\mu\nu]}$.

3. Calculate the nonzero components of the energy-momentum tensor $T^{\mu\nu}$ in cartesian coordinates in an inertial frame in which there is a flat disk of radius r_0 composed of N particles of mass m, rotating counterclockwise in the $x - y$ plane about some fixed point, with fixed (radius-independent) angular velocity ω. Assume that the thickness of the disk is $\ll r_0$, and that N is large enough that one can treat the particles as continuously distributed with fixed number density in the rest frame of the disk. (Notes: (i) Don't worry about what is keeping the particles rotating like this. (ii) Nor should you worry about the effect of their mass on the spacetime – assume it is Minkowski. (iii) Also, you can express your answer using phrases like "inside the disk" and "outside the disk". (iv) Assume that the particle number density n is uniform in the rest frame of the disk. (v) Express you answer in Cartesian coordinates.)

Now suppose there is another such disk present with the same radius and center-of-rotation but with angular velocity $-\omega$, and that the particles do not collide or interact in any way. What is $T^{\mu\nu}$ in this case?

Solution:

Assume the particles are dust-like, in which case the energy-momentum tensor is given by

$$T^{\mu\nu} = p^\mu N^\nu = mnU^\mu U^\nu$$

where m is the mass of each particle, n is the number density in the rest frame of the particles, and U^μ is the four velocity vector field. Situate the disk so that it is centered at x=0, y=0 in the x-y plane. Since geometry is normal, the disk will have a particle number density given by

$$n = \frac{N}{\pi r_o^2 a},$$

if we let a be the thickness of the disk.

The four velocity of a particle is given by

$$U^\mu = \frac{\partial \chi^\mu}{\partial \tau} = \frac{\partial \chi^\mu}{\partial t}\frac{\partial t}{\partial \tau}.$$

We are restricted to the x-y plane, so $U^3 = \frac{dz}{d\tau} = 0$. The time component, U^0, is given by γ, with the 3-velocity magnitude equal to ωr for a particle at radius r. The x and y components, U^1 and U^2, are given by $\gamma \frac{dx}{dt}$ and $\gamma \frac{dy}{dt}$, respectively.

Since this is circular motion, x and y can be parametrized as

$$x = r\cos(\omega t)$$
$$y = r\sin(\omega t)$$

which can be differentiated to yield the velocities

$$v_x = -r\omega \sin(\omega t)$$
$$v_y = r\omega \cos(\omega t).$$

Since U^μ is a velocity field, we need to rewrite the velocity components as a function of position, not time. There is an inherent degeneracy in using sine and cosine alone, so we will use the tangent and solve for t:

$$\frac{y}{x} = \frac{\sin(\omega t)}{\cos(\omega t)} = \tan(\omega t)$$
$$\Rightarrow t = \frac{1}{\omega}\arctan\left(\frac{y}{x}\right)$$

We can then rewrite the velocity equations as

$$v_x = -r\omega \sin(\arctan\left(\frac{y}{x}\right))$$
$$v_y = r\omega \cos(\arctan\left(\frac{y}{x}\right)).$$

We can simplify this further by drawing a right triangle with sides x and y and hypotenuse r, and noticing that

$$\theta = \arctan\left(\frac{y}{x}\right)$$
$$\sin\theta = \sin(\arctan\left(\frac{y}{x}\right)) = \frac{y}{r}$$
$$\cos\theta = \cos(\arctan\left(\frac{y}{x}\right)) = \frac{x}{r}.$$

Thus, our velocities reduce to

$$v_x = -\omega y$$
$$v_y = \omega x$$

and our four velocity field is given by

$$U^\mu = \gamma(1, -\omega y, \omega x, 0).$$

Multiplying out the energy-momentum tensor ($T^{00} = mnU^0U^0, T^{01} = mnU^0U^1$, etc.), we arrive at

$$T^{\mu\nu} = \frac{mN\gamma^2}{\pi r_o a^2} \begin{pmatrix} 1 & -\omega y & \omega x & 0 \\ -\omega y & \omega^2 y^2 & -\omega^2 xy & 0 \\ \omega x & -\omega^2 xy & \omega^2 x^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now we add another disk, with the same number density n, same radius r_o, and opposite angular velocity $-\omega$. The position and velocity of a particle for this ring will be given by

$$x = r\cos(-\omega t) = r\cos(\omega t)$$
$$y = r\sin(-\omega t) = -r\sin(\omega t)$$
$$v_x = -r\omega\sin(\omega t)$$
$$v_y = -r\omega\cos(\omega t).$$

We can solve once again for the time:

$$\frac{y}{x} = \tan(-\omega t) \Rightarrow t = -\frac{1}{\omega}\arctan\left(\frac{y}{x}\right)$$

which makes the velocities become

$$v_x = -r\omega\sin(-\arctan\left(\frac{y}{x}\right)) = \omega y$$
$$v_y = -r\omega\cos(-\arctan\left(\frac{y}{x}\right)) = -\omega x.$$

(We could have obtained this trivially by changing ω to $-\omega$, but it's nice to see that it works out.) The energy-momentum tensor for this ring is then given by

$$T^{\mu\nu} = \frac{mN\gamma^2}{\pi r_o^2 a} \begin{pmatrix} 1 & \omega y & -\omega x & 0 \\ \omega y & \omega^2 y^2 & -\omega^2 xy & 0 \\ -\omega x & -\omega^2 xy & \omega^2 x^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The total energy-momentum tensor for the system is simply the sum of the energy-momentum tensors for each ring:

$$T^{\mu\nu} = T^{\mu\nu}(\omega) + T^{\mu\nu}(-\omega) = \frac{2mN\gamma^2}{\pi r_o^2 a} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega^2 y^2 & -\omega^2 xy & 0 \\ 0 & -\omega^2 xy & \omega^2 x^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that you can see that no energy is flowing anywhere, but that there is (anisotropic) pressure and nonzero shear.