Physics/Astronomy 226, Problem set 3, Due 2/3

1. In class we wrote down a particle number 4-vector

$$N^\mu_{pp} \equiv \sum_n \int d\tau_n\, \delta^4(x^\alpha - x^\alpha_n(\tau_n)) U^\mu_n(\tau_n) \tag{1}$$

for a set of point particles with proper time τ_n, coordinates $x^\alpha_n(\tau_n)$ and 4-velocity $U^\mu_n(\tau_n)$. This is the flux of particle number through a surface of constant x^μ, so that e.g. N^0 is the number density.

(a) Show explicitly that $\partial_\mu N^\mu_{pp} = 0$. That is, act the partial derivative on the expression 1, then pull a bunch of $\delta-$function trickery to show that it vanishes. Explain your trickery clearly. (Hint: do the time integral first to get a δ^3, then show that the time and space parts of the sum cancel.)

(b) We similarly defined a particle energy-momentum tensor

$$T^{\mu\nu}_{pp} \equiv \sum_n \int d\tau_n\, \delta^4(x^\alpha - x^\alpha_n(\tau_n)) \frac{p^\mu_n(\tau_n) p^\nu_n(\tau_n)}{m_n}, \tag{2}$$

where m_n is the mass of the nth particle. Using the same trickery show that

$$\partial_\mu T^{\mu\nu}_{pp} = \sum_n \int d\tau_n \delta^4(x^\alpha - x^\alpha_n(\tau_n)) f^\nu_n(\tau_n),$$

where f_n is the 4-force on the nth particle.

2. A light beam is emitted in vacuo from a height of 10 m and in a direction parallel to the surface of the Earth. Assuming for present purposes that Earth is flat, what is the light beam's distance from Earth after after it travels 1 km? (Use the equivalence principle).

3. Although gravitational time dilation seemed shocking when Einstein first realized it, it's pretty closely tied to the redshift of photons, which is pretty unavoidable. In class we derived the redshift of photons in a gravitational field from the equivalence principle, but it seems that if photons did *not* redshift when going from low- to high-potential, you could build a perpetual motion machine that creates arbitrary amounts of free energy.

(a) Give a reasonably explicit design for such a machine: assume special relativity, Maxwell's EM, quantum mechanics, etc., but assume that photons move through the gravitational potential ϕ with fixed wavelength, and show that you can produce infinite energy from a machine in such a world.

(b) *Extra credit* (i.e. have not tried myself): can you show *quantitatively* for a specific system (or better yet in general, but that's greedy) that the redshift must be given by $\delta\lambda/\lambda = \delta\phi/c^2$ in order to avoid the free-lunch 'problem'?

4. Which of the following are (differentiable) manifolds, and if not, why not:

   (a) The subset of $\Re^2$ satisfying $xy(x^2 + y^2 - 1) = 0$.

   (b) The 2-sphere described in $\Re^3$ by $x^2 + y^2 + z^2 = 1$, where we identify each point (x, y, z) on the sphere with another point $(x, y, -z)$

   (c) The same sphere, but identifying (x, y, z) with $(-x, -y, -z)$.