Physics/Astronomy 226, Problem set 3, Due 2/3
Solutions

1. In class we wrote down a particle number 4-vector

$$N^\mu_{pp} \equiv \sum_n \int d\tau_n \, \delta^4(x^\alpha - x^\alpha_n(\tau_n)) U^\mu_n(\tau_n) \tag{1}$$

for a set of point particles with proper time τ_n, coordinates $x^\alpha_n(\tau_n)$ and 4-velocity $U^\mu_n(\tau_n)$. This is the flux of particle number through a surface of constant x^μ, so that e.g. N^0 is the number density.

(a) Show explicitly that $\partial_\mu N^\mu_{pp} = 0$. That is, act the partial derivative on the expression 1, then pull a bunch of $\delta-$function trickery to show that it vanishes. Explain your trickery clearly. (Hint: do the time integral first to get a δ^3, then show that the time and space parts of the sum cancel.)

(b) We similarly defined a particle energy-momentum tensor

$$T^{\mu\nu}_{pp} \equiv \sum_n \int d\tau_n \, \delta^4(x^\alpha - x^\alpha_n(\tau_n)) \frac{p^\mu_n(\tau_n) p^\nu_n(\tau_n)}{m_n}, \tag{2}$$

where m_n is the mass of the nth particle. Using the same trickery show that

$$\partial_\mu T^{\mu\nu}_{pp} = \sum_n \int d\tau_n \delta^4(x^\alpha - x^\alpha_n(\tau_n)) f^\nu_n(\tau_n),$$

where f_n is the 4-force on the nth particle.

Solution:

(a) First, let's convert our nice covariant expression into a simpler but less covariant-looking one, as we did in class:

$$
\begin{aligned}
\partial_\mu N^\mu_{pp} &= \partial_\mu \sum_n \int d\tau_n \, \delta^4(x^\alpha - x^\alpha_n(\tau_n)) U^\mu_n(\tau_n) \qquad (3) \\
&= \partial_\mu \sum_n \int d\tau_n \, \delta(t - x^0_n(\tau_n)) \delta^3(x^i - x^i_n(\tau_n)) \frac{dx^\mu_n(\tau_n)}{d\tau_n} \\
&= \partial_\mu \sum_n \int \frac{d\tau_n}{dt} dt \, \delta(t - x^0_n(\tau_n)) \delta^3(x^i - x^i_n(\tau_n)) \frac{dx^\mu_n(\tau_n)}{d\tau_n} \\
&= \partial_\mu \sum_n \int dt \, \delta(t - x^0_n(\tau_n)) \delta^3(x^i - x^i_n(\tau_n)) \frac{dx^\mu_n(t(\tau_n))}{dt} \\
&= \partial_\mu \sum_n \delta^3(x^i - x^i_n(t)) \frac{dx^\mu_n(t)}{dt} \\
&= \partial_\mu \sum_n \delta^3(x^i - x^i_n(t)) \frac{dx^\mu_n(t)}{dt} \\
&= \partial_t \sum_n \delta^3(x^i - x^i_n(t)) \frac{dx^0_n(t)}{dt} + \partial_j \sum_n \delta^3(x^i - x^i_n(t)) \frac{dx^j_n(t)}{dt}
\end{aligned}
\qquad (4)
$$

Now, let's look at the first term:

$$
\begin{aligned}
&= \sum_n \left[\delta^3(x^i - x^i_n(t)) \frac{d}{dt} 1 + \partial_t \delta^3(x^i - x^i_n(t)) \right] \qquad (5) \\
&= \sum_n \frac{dx^j_n}{dt} \frac{\partial}{\partial x^j_n} \delta^3(x^i - x^i_n(t))
\end{aligned}
$$

Now, because

$$
\frac{\partial}{\partial x^i_n} \delta^3(x^i - x^i_n(t)) = -\frac{\partial}{\partial x^i} \delta^3(x^i - x^i_n(t)),
$$

we can change the j to an i summation index in this equation, and then it cancels with the second term of Eq. 7.

(b) Here, the procedure is essentially the same, except that we carry around an P^μ:

$$\partial_\mu T^{\mu\nu}_{pp} = \partial_\mu \sum_n \int d\tau_n \, P^\nu_n(\tau_n)\delta^4(x^\alpha - x^\alpha_n(\tau_n))U^\mu_n(\tau_n) \tag{6}$$

$$\begin{aligned}
&= \partial_\mu \sum_n \int d\tau_n \, \delta(t - x^0_n(\tau_n))\delta^3(x^i - x^i_n(\tau_n))\frac{dx^\mu_n(\tau_n)}{d\tau_n}P^\nu_n(\tau_n) \\
&= \partial_\mu \sum_n \int \frac{d\tau_n}{dt} dt \, \delta(t - x^0_n(\tau_n))\delta^3(x^i - x^i_n(\tau_n))\frac{dx^\mu_n(\tau_n)}{d\tau_n}P^\nu_n(\tau_n) \\
&= \partial_\mu \sum_n \int dt \, \delta(t - x^0_n(\tau_n))\delta^3(x^i - x^i_n(\tau_n))\frac{dx^\mu_n(t(\tau_n))}{dt}P^\nu_n(\tau_n) \\
&= \partial_\mu \sum_n \delta^3(x^i - x^i_n(t))\frac{dx^\mu_n(t)}{dt}P^\nu_n(t) \\
&= \partial_\mu \sum_n \delta^3(x^i - x^i_n(t))\frac{dx^\mu_n(t)}{dt}P^\nu_n(t) \\
&= \partial_t \sum_n \delta^3(x^i - x^i_n(t))\frac{dx^0_n(t)}{dt}P^\nu_n(t) + \partial_j \sum_n \delta^3(x^i - x^i_n(t))\frac{dx^j_n(t)}{dt}P^\nu_n(t)
\end{aligned} \tag{7}$$

Now, though, the ∂_t in the first term spits out a second part that does not cancel:

$$\partial_\mu T^{\mu\nu}_{pp} = \sum_n \delta^3(x^i - x^i_n(t))\frac{dP^\nu_n(t)}{dt} = \sum_n \delta^3(x^i - x^i_n(t))f^\nu_n(t).$$

Then, to get the desired expression we just reverse the initial steps to 'add back in' the $\int d\tau$ and recover the covariant-looking expression

$$\partial_\mu T^{\mu\nu}_{pp} = \sum_n \int d\tau_n \delta^4(x^\alpha - x^\alpha_n(\tau_n))f^\nu_n(\tau_n),$$

2. A light beam is emitted in vacuo from a height of 10 m and in a direction parallel to the surface of the Earth. Assuming for present purposes that Earth is flat, what is the light beam's distance from Earth after after it travels 1 km? (Use the equivalence principle).

   Solution: The time taken by the light to travel a kilometer is given by:

$$\triangle t = \frac{10^3 \ m}{c} = \frac{10^3 \ m}{3 \cdot 10^8 \ m/s} = \frac{1}{3} \cdot 10^{-5} \ sec$$

   And so, the distance moved by the freely falling frame is:

$$\triangle y = \frac{1}{2} \, a \, \triangle t^2 = \frac{1}{2} \ (9.8 m/s^2) \cdot (\frac{1}{3} \cdot 10^{-5} \ sec)^2 = 5.44 \cdot 10^{-11} m$$

   Yielding a final height of the light beam above the Earth's surface of:

$$Y = 10m - 5.44 \cdot 10^{-11} m$$

3. Although gravitational time dilation seemed shocking when Einstein first realized it, it's pretty closely tied to the redshift of photons, which is pretty unavoidable. In class we derived the redshift of photons in a gravitational field from the equivalence principle, but it seems that if photons did *not* redshift when going from low- to high-potential, you could build a perpetual motion machine that creates arbitrary amounts of free energy.

   (a) Give a reasonably explicit design for such a machine: assume special relativity, Maxwell's EM, quantum mechanics, etc., but assume that photons move through the gravitational potential ϕ with fixed wavelength, and show that you can produce infinite energy from a machine in such a world.

   (b) *Extra credit* (i.e. have not tried myself): can you show *quantitatively* for a specific system (or better yet in general, but that's greedy) that the redshift must be given by $\delta\lambda/\lambda = \delta\phi/c^2$ in order to avoid the free-lunch 'problem'?

   Solution:

   (a) I got a lot of pretty vague answers for this one, and very few patentable ideas. It's pretty clear that to get the infinite energy source, you need to send photons up the gravitational potential, then convert the photon's energy to another form of energy (like rest mass) that we know picks up energy as we go back down the potential. There were various ideas for this.

   A simple one, adopted from some of yours, I will call the 'happy fun energy ball' (HFEB). It has rest mass m and different energy levels, and in particular it can absorb a photon to go from energy level E_1 to E_2 using photon energy ΔE_{12}. (For simplicity let's say this is the smallest increment of energy it can absorb.) It also bounces with near-perfect elasticity. But when it bounces, it gives off two photons, one of energy $\tilde{E}$ and one of energy ΔE_{12}. Now, we sit this in a mirrored room and let it bounce (we could use a bunch of them if we like). It starts to give off photon pairs. But whenever it absorbs a ΔE_{12} photon, it descends the gravity well with a bit more rest mass, $m + \Delta E_{12}$, so it picks up an extra $\Delta E_{12} g \delta h$ in energy relative to what it lost on the ascent, where δh is the height difference between he top of its trajectory and where it absorbed the photon. When it bounces, it gives off a ΔE_{12} photon completing the cycle except for this extra energy, which comes out in the $\tilde{E}$ photon. So all of the $\tilde{E}$ photons are effectively free energy.

   (b) I think I'm going to defer this one until later in the quarter after we've talked about conserved quantities in GR.

4. Which of the following are (differentiable) manifolds, and if not, why not:

   (a) The subset of $\Re^2$ satisfying $xy(x^2 + y^2 - 1) = 0$.

   (b) The 2-sphere described in $\Re^3$ by $x^2 + y^2 + z^2 = 1$, where we identify each point (x, y, z) on the sphere with another point $(x, y, -z)$

   (c) The same sphere, but identifying (x, y, z) with $(-x, -y, -z)$.

   Solution:

(a) The subset of $\Re^2$ that satisfies $xy(x^2 + y^2 - 1) = 0$ includes the three solutions to that equation: the lines $x = 0$ and $y = 0$, and the circle $x^2 + y^2 = 1$. It is not a valid differentiable manifold because of the 5 junction points between the solutions (x=0 and y=0, x=0 and y=±1, y=0 and x=±1).

(b) The 2-sphere in $\Re^3$, with each point (x,y,z) on the sphere identified with another point (x,y,-z), is not a manifold because it has a boundary at z=0. The points at z=0 are mapped onto themselves. This means that this little region can be mapped to the half plane $z \geq 0$, but *not* to an open subset of $\mathcal{R}^N$. It is a *manifold with boundary* but not a manifold.

(c) The 2-sphere in $\Re^3$, with each point (x,y,z) on the sphere identified with another point (-x,-y,-z) is fundamentally different from part (b) in that no point is identified onto itself. Therefore every local section on the sphere can be mapped to an open subset of $\Re^3$, ie. there is no boundary on the manifold.