Physics/Astronomy 226, Problem set 5, Due 2/17 Solutions

Reading: Ch. 3, start Ch. 4

1. Prove that $R_{\lambda\nu\alpha\beta} + R_{\lambda\alpha\beta\nu} + R_{\lambda\beta\nu\alpha} = 0.$

Solution:

We can evaluate this in a local inertial frame, in which the Christoffel symbols (but not their derivatives) vanish, so that:

$$R_{\lambda\nu\alpha\beta} = g_{\lambda\rho} (\partial_{\alpha} \Gamma^{\rho}_{\beta\nu} - \partial_{\beta} \Gamma^{\rho}_{\alpha\nu}) \tag{1}$$

$$= \frac{1}{2} (\partial_{\alpha} \partial_{\nu} g_{\lambda\beta} - \partial_{\alpha} \partial_{\lambda} g_{\beta\nu} - \partial_{\beta} \partial_{\nu} g_{\lambda\alpha} + \partial_{\beta} \partial_{\lambda} g_{\alpha\nu}).$$
(2)

Under the permutation of indicies $\nu \to \alpha \to \beta \to \nu$,

$$R_{\lambda\alpha\beta\nu} = \frac{1}{2} (\partial_{\beta}\partial_{\alpha}g_{\lambda\nu} - \partial_{\beta}\partial_{\lambda}g_{\nu\alpha} - \partial_{\nu}\partial_{\alpha}g_{\lambda\beta} + \partial_{\nu}\partial_{\lambda}g_{\beta\alpha}).$$
(3)

Under another permutation of $\nu \to \alpha \to \beta \to \nu$,

$$R_{\lambda\beta\nu\alpha} = \frac{1}{2} (\partial_{\nu}\partial_{\beta}g_{\lambda\alpha} - \partial_{\nu}\partial_{\lambda}g_{\alpha\beta} - \partial_{\alpha}\partial_{\beta}g_{\lambda\nu} + \partial_{\alpha}\partial_{\lambda}g_{\nu\beta}).$$
(4)

Taking the sum of the terms,

$$R_{\lambda[\nu\alpha\beta]} = \frac{1}{2} [(\partial_{\alpha}\partial_{\nu}g_{\lambda\beta} - \partial_{\nu}\partial_{\alpha}g_{\lambda\beta}) + (\partial_{\beta}\partial_{\alpha}g_{\lambda\nu} - \partial_{\alpha}\partial_{\beta}g_{\lambda\nu}) + (\partial_{\nu}\partial_{\beta}g_{\lambda\alpha} - \partial_{\beta}\partial_{\nu}g_{\lambda\alpha}) + \partial_{\alpha}\partial_{\lambda}(g_{\nu\beta} - g_{\beta\nu}) + \partial_{\beta}\partial_{\lambda}(g_{\alpha\nu} - g_{\nu\alpha}) + \partial_{\nu}\partial_{\lambda}(g_{\beta\alpha} - g_{\alpha\beta})].$$
(5)

The first three terms vanish because the partial derivatives commute, and the last three terms vanish because the metric $g_{\mu\nu}$ is symmetric. Therefore,

$$R_{\lambda\nu\alpha\beta} + R_{\lambda\alpha\beta\nu} + R_{\lambda\beta\nu\alpha} = R_{\lambda[\nu\alpha\beta]} = 0.$$
(6)

Because this is a tensorial equation, and is true in our locally inertial coordinates, it is always true.

2. Picture a donut in 3d Euclidean space. It may be either chocolate or glazed, as long as when viewed from the top it looks like two concentric circles of radius r_1 and $r_2 > r_1$. Let $b = (r_1 + r_2)/2$ and $a = (r_2 - r_1)/2$.

- (a) Set up coordinates θ , ϕ on the donut surface (for consistency, let θ label the angle about the center of the donut as measured from above, and ϕ measure the angle around a circular cross-section of the donut.)
- (b) Write down the metric g_{ij} this surface inherits from the Euclidean space it is embedded in.
- (c) Compute all non-vanishing connection coefficients $\Gamma^u_{\alpha\beta}$.
- (d) Compute all nonzero components of $R_{\mu\nu\alpha\beta}$, $R_{\mu\nu}$, and R.

Solution:

Given a torus with outer radius R and inner radius r, we make the assignments a = (R - r)/2 and b = (R + r)/2. We embed it in \mathbb{R}^3 with cylindrical coordinates r, θ, z , such that:

$$z = a \sin(\phi)$$

$$r = b + a \cos(\phi)$$

$$\theta = \theta.$$
(7)

The metric is then:

$$ds^{2} = dz^{2} + dr^{2} + r^{2}d\theta^{2} = a^{2}d\phi^{2} + (b + a\cos(\phi))^{2}d\theta^{2}$$
(8)

or,

$$g_{\mu\nu} = \begin{pmatrix} a^2 & 0\\ 0 & (b+a\cos(\phi))^2 \end{pmatrix}$$
(9)

and

$$g^{\mu\nu} = \begin{pmatrix} a^{-2} & 0\\ 0 & (b+a\cos(\phi))^{-2} \end{pmatrix}.$$
 (10)

We start by finding the non-zero connection coefficients. For upper index θ , we must evaluate $\Gamma^{\theta}_{\theta\theta}, \Gamma^{\theta}_{\phi\phi}, \Gamma^{\theta}_{\theta\phi}$ and $\Gamma^{\theta}_{\phi\phi}$. Given that

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}g^{\sigma\sigma}(\partial_{\mu}g_{\nu\sigma} + \partial_{\nu}g_{\sigma\mu} - \partial_{\sigma}g_{\mu\nu}), \qquad (11)$$

we have:

$$\begin{aligned}
 \Gamma^{\theta}_{\theta\theta} &= 0 \\
 \Gamma^{\theta}_{\phi\phi} &= 0 \\
 \Gamma^{\theta}_{\phi\theta} &= \frac{1}{2}g^{\theta\theta}\partial_{\phi}g_{\theta\theta} \\
 &= \frac{-a\sin(\phi)}{(b+a\cos(\phi))} \\
 \Gamma^{\theta}_{\theta\phi} &= \Gamma^{\theta}_{\phi\theta}.
 \end{aligned}$$
(12)

For upper index ϕ , we evaluate the terms $\Gamma^{\phi}_{\phi\phi}, \Gamma^{\phi}_{\theta\phi}, \Gamma^{\phi}_{\phi\theta}$ and $\Gamma^{\phi}_{\theta\theta}$.

$$\Gamma^{\phi}_{\phi\phi} = 0$$

$$\Gamma^{\phi}_{\theta\phi} = 0$$

$$\Gamma^{\phi}_{\theta\theta} = -\frac{1}{2}g^{\phi\phi}\partial_{\phi}g_{\theta\theta}$$

$$= \frac{\sin(\phi)}{a}(b + a\cos(\phi))$$

$$\Gamma^{\phi}_{\phi\theta} = 0.$$
(13)

We now want to find the non-zero components of the curvature tensor. Before evaluating a component at random, let us consider these properties:

- If n = 2 and there are $\frac{1}{12}n^2(n^2 1)$ independent components of the tensor, we only expect one independent component overall.
- $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$ so a non-zero component must have its first two indicies distinct (i.e. one must be θ and the other ϕ).
- $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$, so the last two indicies must also be distinct for the component to be non-zero.

Therefore, we conclude that $R_{\theta\phi\theta\phi}$ must be non-zero. It's easiest to compute it with one raised index:

$$\begin{aligned}
R^{\rho}_{\sigma\mu\nu} &= \partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \\
\Rightarrow R^{\theta}_{\ \phi\theta\phi} &= -\partial_{\phi}\Gamma^{\theta}_{\theta\phi} - (\Gamma^{\theta}_{\phi\theta})^{2} \\
&= -\partial_{\phi}(\frac{-a\sin(\phi)}{(b+a\cos(\phi))}) - (\frac{-a\sin(\phi)}{(b+a\cos(\phi))})^{2} \\
&= (\frac{a\cos(\phi)}{b+a\cos(\phi)}).
\end{aligned}$$
(14)

Lowering the index by contraction:

$$R_{\theta\phi\theta\phi} = g_{\theta\lambda} R^{\lambda}_{\ \phi\theta\phi} = g_{\theta\theta} R^{\theta}_{\ \phi\theta\phi} = a\cos(\phi)(b + a\cos(\phi)).$$
(15)

All the other non-zero components can be found by symmetry.

$$R_{\phi\theta\phi\phi} = -R_{\phi\theta\phi\phi} = -R_{\theta\phi\phi\phi} = R_{\theta\phi\phi\phi} \tag{16}$$

We find the Ricci tensor by contraction, $R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu}$, so that:

$$R_{\theta\theta} = g^{\phi\phi} R_{\phi\theta\phi\theta}$$

$$= \frac{\cos(\phi)}{(b+a\cos(\phi))}$$

$$R_{\phi\phi} = g^{\theta\theta} R_{\theta\phi\theta\phi}$$

$$= \frac{a\cos(\phi)}{b+a\cos(\phi)}.$$
(17)

The Ricci scalar R is also found by contraction, where $R = R^{\mu}_{\mu} = g^{\mu\nu}R_{\mu\nu}$. Therefore,

$$R = g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi}$$

= $\frac{2\cos(\phi)}{a(b+a\cos(\phi))}$. (18)

FIG. 1: Torus (figure stolen from Mathworld website).

It is worth noticing two things about this problem. Firstly, the torus is not maximally symmetric as defined by Carroll, i.e. there does not exist a constant α such that $R_{\rho\sigma\mu\nu} = \alpha^{-2}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu})$. We might have suspected this, since the torus doesn't have as many symmetries as the space in which it is embedded. Secondly, if one ignores the pesky word "embedded" in the problem set, one can simply slice open the torus and put it in lovely flat \mathbf{R}^2 where all connection coefficients and components of the curvature tensor are zero!

3. The donut hole left over from the above donut has a roughly spherical surface. A sphere with coordinates (θ, ϕ) has metric

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2.$$

- (a) Show that lines of constant ϕ (longitude) are geodesics, and that the only line of constant θ (latitude) that is a geodesic is the $\theta = \pi/2$ (the equator).
- (b) Take a vector with components $V^{\mu} = (V^{\theta}, V^{\phi}) = (1, 0)$ and parallel-transport it once around a circle of constant latitude, $\theta = \theta_0$. What are the components of the resulting vector, as a function of θ_0 ?

Solution:

Consider a 2-sphere with the metric:

$$ds^2 = d\theta^2 + \sin^2\theta \, d\phi$$

a) We must first compute the connection coefficients, given by:

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2g_{\lambda\lambda}} \left(\partial_{\mu}g_{\nu\lambda} + \partial_{\nu}g_{\lambda\mu} - \partial_{\lambda}g_{\mu\nu} \right)$$

The only nonzero terms will come from $\partial_{\theta} g_{\phi\phi}$. The non-zero connection coefficients are therefore:

$$\Gamma^{\phi}{}_{\theta\phi} = \frac{1}{2g_{\phi\phi}} \partial_{\theta}g_{\phi\phi} = \frac{1}{2\sin\theta} \partial_{\theta}(\sin^2\theta) = \cot\theta$$
$$\Gamma^{\phi}{}_{\phi\theta} = \frac{1}{2g_{\phi\phi}} \partial_{\theta}g_{\phi\phi} = \cot\theta$$
$$\Gamma^{\theta}{}_{\phi\phi} = \frac{-1}{2g_{\theta\theta}} \partial_{\theta}g_{\phi\phi} = \frac{1}{2} \partial_{\theta}(\sin^2\theta) = -\sin\theta\cos\theta$$

We may now look at the geodesic equation:

$$\frac{d^2 x^{\mu}}{d\tau^2} + \Gamma^{\mu}{}_{\rho\sigma} \frac{dx^{\rho}}{d\tau} \frac{dx^{\sigma}}{d\tau} = 0$$

For $\mu = \phi$, we have:

$$\frac{d^2x^{\phi}}{d\tau^2} + \Gamma^{\phi}{}_{\theta\phi}\frac{dx^{\theta}}{d\tau}\frac{dx^{\phi}}{d\tau} + \Gamma^{\phi}{}_{\phi\theta}\frac{dx^{\phi}}{d\tau}\frac{dx^{\theta}}{d\tau} = 0$$

Substituting for the connection coefficients:

$$\frac{d^2\phi}{d\tau^2} + 2\cot\theta \frac{d\phi}{d\tau} \frac{d\theta}{d\tau} = 0$$
(19)

For $\mu = \theta$, we have:

$$\frac{d^2x^{\theta}}{d\tau^2} + \Gamma^{\theta}{}_{\phi\phi}\frac{dx^{\phi}}{d\tau}\frac{dx^{\phi}}{d\tau} = 0 = \frac{d^2\theta}{d\tau^2} - \sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2 \tag{20}$$

It can be seen that if ϕ were constant, then geodesic equation (7) will be satisfied trivially, and Eq. (8) will be satisfied if τ is linearly related to θ .

If θ were constant, then Eq. (7) is likewise satisfied for τ linearly related to ϕ . But geodesic equation (8) is:

$$\sin\theta_c\cos\theta_c\left(\frac{d\phi}{d\tau}\right)^2 = 0$$

This will be satisfied while varying ϕ only if the constant θ_c is $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$The choices $\theta_c = 0, \pi, 2\pi$ are uninteresting because they represent motion on the poles, so varying ϕ does nothing. The choices $\theta_c = \frac{\pi}{2}, \frac{3\pi}{2}$ correspond to motion along the equator.

b) Consider a vector with components $V^{\mu} = (V^{\theta}, V^{\phi}) = (1, 0)$. If the vector is parallel propagated around a circle of constant $\theta = \theta_c$, it must satisfy the equation of parallel transport along its path:

$$\frac{dV^{\mu}}{d\lambda} + {\Gamma^{\mu}}_{\sigma\rho} \frac{dx^{\sigma}}{d\lambda} V^{\rho} = 0$$

For $\mu = \theta$ and $\mu = \phi$, this becomes:

$$\frac{dV^{\theta}}{d\lambda} + \Gamma^{\theta}{}_{\phi\phi}\frac{dx^{\phi}}{d\lambda}V^{\phi} = \frac{dV^{\theta}}{d\lambda} - \sin\theta_{o}\cos\theta_{o}\frac{d\phi}{d\lambda}V^{\phi} = 0$$
$$\frac{dV^{\phi}}{d\lambda} + \Gamma^{\phi}{}_{\phi\theta}\frac{dx^{\phi}}{d\lambda}V^{\phi} = \frac{dV^{\phi}}{d\lambda} + \cot\theta_{o}\frac{d\phi}{d\lambda}V^{\theta} = 0$$

Re-writing $\frac{dV}{d\lambda}$ as $\frac{dV}{d\phi}\frac{d\phi}{d\lambda}$, we have 2 coupled 1st order equations to integrate:

$$\frac{dV^{\theta}}{d\phi} - \sin\theta_o \cos\theta_o V^{\phi} = 0$$
$$\frac{dV^{\phi}}{d\phi} + \cot\theta_o V^{\theta} = 0$$

These decouple into two 2nd order equations:

$$\frac{d^2 V^{\phi}}{d\phi^2} + \cos^2 \theta_o V^{\phi} = 0$$
$$\frac{d^2 V^{\theta}}{d\phi^2} + \cos^2 \theta_o V^{\theta} = 0$$

The general solution to these is:

$$V^{\phi} = A\cos(\phi\cos\theta_o) + B\sin(\phi\cos\theta_o)$$
$$V^{\theta} = C\cos(\phi\cos\theta_o) + D\sin(\phi\cos\theta_o)$$

We can solve for the constants using the initial conditions on V and $\frac{dV}{d\phi}$ (found by evaluating the coupled 1st order equations above at $\phi = 0$):

$$(V^{\theta}(0), V^{\phi}(0)) = (1, 0)$$

$$\Rightarrow A = 0 \text{ and } C = 1$$

$$\left(\frac{dV^{\theta}}{d\phi}\Big|_{\phi=0}, \frac{dV^{\phi}}{d\phi}\Big|_{\phi=0}\right) = (0, -\cot\theta_o)$$

$$\Rightarrow B = -\frac{1}{\sin\theta_o} \text{ and } D = 0$$

Therefore, $V(\phi = 2\pi)$ is given by:

$$(V^{\theta}, V^{\phi}) = \left(\cos\left[2\pi\cos\theta_o\right], -\frac{\sin\left[2\pi\cos\theta_o\right]}{\sin\theta_o}\right)$$

If $\theta_o = \frac{\pi}{2}$ (parallel transport along the equitorial geodesic), then we get back our original vector at $\phi = 2\pi$ as expected. Also, the norm of V is equal to one as it should be:

$$g_{\mu\nu}V^{\mu}V^{\nu} = g_{\theta\theta}V^{\theta}V^{\theta} + g_{\phi\phi}(\theta_o)V^{\phi}V^{\phi} = \cos^2\left[\phi\cos\theta_o\right] + \sin^2\theta_o\frac{\sin^2\left[\phi\cos\theta_o\right]}{\sin^2\theta_o} = 1$$