Physics/Astronomy 226, Problem set 6, Due 2/24

Reading: Ch. 4

1. In flat spacetime, Maxwell's equations can be written

$$\partial_\nu F^{\mu\nu} = J^\mu,$$

where $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$. In going to curved spacetime according to the Equivalence Principle, we would like to replace partial derivatives such as ∂_μ with covariant ones like ∇_μ.

   (a) Show that this procedure is ambiguous, i.e. that there are two inequivalent ways of making this substitution.

   (b) Express one set of equations in terms of the others and the Ricci tensor $R_{\mu\nu}$.

   (c) In which if either of your two equations is current covariantly conserved, i.e. does:

$$\nabla_\mu J^\mu = 0?$$

2. Fill in some of the details of the 'GR weak field' calculation I did in class:

Assume spacetime is "nearly flat" in the sense that coordinates can be found for which

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \text{ where } |h_{\mu\nu}| \ll 1.$$

We will then raise and lower indices on tensors using $\eta_{\mu\nu}$ and its inverse, and only go to first order in $h_{\mu\nu}$ in all calculations.

   (a) Write down $\Gamma^\delta_{\alpha\beta}$, $R_{\alpha\beta}$, and R to first order in h, and let $\bar{h}_{\alpha\beta} \equiv h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h$ and $h \equiv h^\mu{}_\mu$.

   (b) What are $\bar{h}^{\alpha\beta}$ and $\bar{h}$?

   (c) Show then that $G_{\alpha\beta}$ takes the form given in class.

   (d) Show that under the coord. transform

$$x^{\alpha'} = x^\alpha - \epsilon\xi^\alpha(x),$$

   the components of R^α_β, and thus also R, are unchanged. Here, $\epsilon \ll 1$ is fixed, and ξ^μ is some vector field.

   (e) Find the behavior of $\bar{h}'_{\alpha\beta}$ under this coordinate transformation, and show that we can find a $\xi^\mu(x)$ such that under such a coordinate transformation,

$$\partial^\alpha \bar{h}'_{\alpha\beta} = 0.$$

   (f) Show then that in these coordinates,

$$G_{\mu\nu} = -\frac{1}{2}\Box\bar{h}'_{\mu\nu}.$$

3. In class we mentioned the energy-momentum tensor for a point-particle following worldline $y^\alpha(\tau)$:

$$T^{\mu\nu}_{pp} = m \int d\tau \frac{1}{\sqrt{-g}} \delta^4(x^\alpha - y^\alpha(\tau)) U^\mu U^\nu. \tag{1}$$

   (a) Show that in general,

$$\nabla_\mu T^{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} T^{\mu\nu}) + \Gamma^\nu_{\alpha\beta} T^{\alpha\beta},$$

   where U is the 4-velocity as usual.

   (b) Use this to show that

$$\nabla_\mu T^{\mu\nu}_{pp} = 0$$

   implies that the $y^\alpha(\tau)$ obeys the geodesic equation. Cool, huh?

   (Hint: plug part a into part b, and convert the $x-$derivative to a y-derivative. Then integrate by parts.)

   (c) We saw in class that the action

$$S_{pp} = m \int d\tau = m \int d\tau \left[-g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right]^{1/2}$$

   gave the geodesic equation as an equation of motion for $x^\mu(\tau)$.

   As discussed in class, we can obtain the energy-momentum tensor by varying the action with respect to the metric. Show that

$$\frac{-2}{\sqrt{-g}} \frac{\delta S_{pp}}{\delta g^{\mu\nu}} = T_{\mu\nu,pp},$$

   with $T_{\mu\nu,pp}$ as given in Eq. 1