Physics/Astronomy 226, Problem set 6, Due 2/24
Solutions

Reading: Ch. 4

1. In flat spacetime, Maxwell's equations can be written

$$\partial_\nu F^{\mu\nu} = J^\mu,$$

where $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$. In going to curved spacetime according to the Equivalence Principle, we would like to replace partial derivatives such as ∂_μ with covariant ones like ∇_μ.

   (a) Show that this procedure is ambiguous, i.e. that there are two inequivalent ways of making this substitution.

   (b) Express one set of equations in terms of the others and the Ricci tensor $R_{\mu\nu}$.

   (c) In which if either of your two equations is current covariantly conserved, i.e. does:

$$\nabla_\mu J^\mu = 0?$$

Solution:

(a) To show that making the equation $\partial_\nu F^{\mu\nu} = J^\mu$ covariant is ambiguous, we need to use the fact that partial derivatives like ∂_ν commute, while covariant derivatives like ∇_ν do not. The first step is to write $F^{\mu\nu}$ in terms of A:

$$\partial_\nu F^{\mu\nu} = 2\partial_\nu \partial^{[\mu}A^{\nu]} = \partial_\nu \partial^\mu A^\nu - \partial_\nu \partial^\nu A^\mu.$$

This can also be written as

$$\partial^\mu \partial_\nu A^\nu - \partial_\nu \partial^\nu A^\mu$$

since the partial derivatives commute. However, since covariant derivatives do not commute,

$$\nabla_\nu \nabla^\mu A^\nu - \nabla_\nu \nabla^\nu A^\mu \neq \nabla^\mu \nabla_\nu A^\nu - \nabla_\nu \nabla^\nu A^\mu$$

(b) The difference between the two expressions will be

$$\nabla_\nu \nabla^\mu A^\nu - \nabla^\mu \nabla_\nu A^\nu$$

If we recall from chapter 3 that (for a torsionless metric) the commutator of two covariant derivatives acting on a vector is

$$[\nabla_\mu, \nabla_\nu] V^\rho = \nabla_\mu \nabla_\nu V^\rho - \nabla_\nu \nabla_\mu V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma$$

We can see that (if we lower the μ) the difference between the two expressions is just

$$R^\nu{}_{\lambda\nu\mu} A^\lambda = R_{\lambda\mu} A^\lambda = R_{\mu\nu} A^\nu$$

Re-raising the μ, we see that

$$\nabla_\nu \nabla^\mu A^\nu - \nabla_\nu \nabla^\nu A^\mu = \nabla^\mu \nabla_\nu A^\nu - \nabla_\nu \nabla^\nu A^\mu + R^\mu{}_\nu A^\nu$$

(c) For $\nabla_\mu J^\mu = 0$, we thus need either:

$$(a)\quad \nabla_\mu \nabla_\nu \nabla^\mu A^\nu - \nabla_\mu \nabla_\nu \nabla^\nu A^\mu = 0$$

or

$$(b)\quad \nabla_\mu \nabla^\mu \nabla_\nu A^\nu - \nabla_\mu \nabla_\nu \nabla^\nu A^\mu = 0$$

Now, they can't both be zero, because going to a LIF, we know that the difference,

$$\nabla^\mu R_{\mu\nu} A^\nu = \frac{1}{2} \nabla_\nu R A^\nu$$

has no particular reason to vanish. But also, (a) reads (swapping $\mu \leftrightarrow \nu$ in the first term):

$$\begin{aligned}
&= \nabla_\nu \nabla_\mu \nabla^\nu A^\mu - \nabla_\mu \nabla_\nu \nabla^\nu A^\mu \\
&= [\nabla_\nu, \nabla_\mu] \nabla^\nu A^\mu \\
&= R^\nu{}_{\lambda\nu\mu} \nabla^\lambda A^\mu + R^\mu{}_{\lambda\nu\mu} \nabla^\nu A^\lambda \\
&= R_{\lambda\mu} \nabla^\lambda A^\mu - R_{\lambda\nu} \nabla^\nu A^\lambda \\
&= 0
\end{aligned} \tag{1}$$

so the current is conserved in version (a) but not (b).

2. Fill in some of the details of the 'GR weak field' calculation I did in class:

Assume spacetime is "nearly flat" in the sense that coordinates can be found for which

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \text{ where } |h_{\mu\nu}| \ll 1.$$

We will then raise and lower indices on tensors using $\eta_{\mu\nu}$ and its inverse, and only go to first order in $h_{\mu\nu}$ in all calculations.

(a) Write down $\Gamma^\delta_{\alpha\beta}$, $R_{\alpha\beta}$, and R to first order in h, and let $\bar{h}_{\alpha\beta} \equiv h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} h$ and $h \equiv h^\mu{}_\mu$.

(b) What are $\bar{h}^{\alpha\beta}$ and $\bar{h}$?

(c) Show then that $G_{\alpha\beta}$ takes the form given in class.

(d) Show that under the coord. transform

$$x^{\alpha'} = x^\alpha - \epsilon \xi^\alpha(x),$$

the components of R^α_β, and thus also R, are unchanged. Here, $\epsilon \ll 1$ is fixed, and ξ^μ is some vector field.

(e) Find the behavior of $\bar{h}'_{\alpha\beta}$ under this coordinate transformation, and show that we can find a $\xi^\mu(x)$ such that under such a coordinate transformation,

$$\partial^\alpha \bar{h}'_{\alpha\beta} = 0.$$

(f) Show then that in these coordinates,

$$G_{\mu\nu} = -\frac{1}{2}\Box \bar{h}'_{\mu\nu}.$$

Solution:

(a)

$$\begin{aligned} \Gamma^{\delta}_{\alpha\beta} &= \frac{1}{2}\left(\eta^{\delta\sigma} - h^{\delta\sigma}\right)\left(\partial_\alpha h_{\beta\sigma} + \partial_\beta h_{\sigma\alpha} - \partial_\sigma h_{\alpha\beta}\right) \\ &= \frac{1}{2}\eta^{\delta\sigma}\left(\partial_\alpha h_{\beta\sigma} + \partial_\beta h_{\sigma\alpha} - \partial_\sigma h_{\alpha\beta}\right) + O(h^2). \end{aligned} \tag{2}$$

For R, we note that terms with products of Γs will be second order in h, so we have

$$R_{\alpha\beta} = \partial_\lambda \Gamma^\lambda_{\alpha\beta} - \partial_\beta \Gamma^\lambda_{\alpha\lambda} \tag{3}$$

$$= \frac{1}{2}\left[\partial^\sigma \partial_\alpha h_{\beta\sigma} - \partial^\sigma \partial_\sigma h_{\beta\alpha} - \partial_\beta \partial_\alpha h^\sigma{}_\sigma + \partial_\beta \partial^\lambda h_{\alpha\lambda}\right] \tag{4}$$

This gives

$$R = \partial^\sigma \partial^\beta h_{\beta\sigma} - \partial_\sigma \partial^\sigma h.$$

(b)

$$\bar{h}^{\alpha\beta} = \eta^{\alpha\nu}\eta^{\beta\mu}(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h) \tag{5}$$

$$= h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta}h. \tag{6}$$

so

$$\bar{h} = h - \frac{1}{2}\cdot 4 \cdot h = -h.$$

(c)

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h$$

which can be converted to the form given in class.

(d) We have

$$\delta^{\alpha'}_{\alpha} = 1 - \epsilon\frac{\partial \xi^{\alpha'}}{\partial x^\alpha},$$

and to first order in ϵ,

$$\delta^{\alpha}_{\alpha'} = 1 + \epsilon\frac{\partial \xi^{\alpha}}{\partial x^{\alpha'}}$$

Then:

$$R^{\alpha'}{}_{\beta'} = \left(1 + \epsilon\frac{\partial \xi^\alpha}{\partial x^{\alpha'}}\right)\left(1 - \epsilon\frac{\partial \xi^{\beta'}}{\partial x^\beta}\right)R^\alpha{}_\beta = R^\alpha{}_\beta + O(\epsilon^2)$$

(e) Going through the calculation yields:

$$\partial^\nu \bar{h}'_{\mu\nu} = \partial^\alpha \bar{h}_{\mu\nu} + \partial_\nu \partial^\nu \xi^\mu.$$

Thus we can take the equation $\partial_\nu \partial^\nu = -\partial^\alpha \bar{h}_{\mu\nu}$ and notice that it is a wave equation with a source $-\partial^\alpha \bar{h}_{\mu\nu}$. This can be solved using Green functions to obtain a ξ^μ for which $\partial^\nu \bar{h}'_{\mu\nu} = 0$.

(f) Given the commutation of partial derivatives, there are three terms in the expression for $G_{\mu\nu}$ that vanish in our new gauge, leaving just the term in $\Box h$.

3. In class we mentioned the energy-momentum tensor for a point-particle following worldline $y^\alpha(\tau)$:

$$T^{\mu\nu}_{pp} = m \int d\tau \frac{1}{\sqrt{-g}} \delta^4(x^\alpha - y^\alpha(\tau)) U^\mu U^\nu. \tag{7}$$

(a) Show that in general,

$$\nabla_\mu T^{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} T^{\mu\nu}) + \Gamma^\nu_{\alpha\beta} T^{\alpha\beta},$$

where U is the 4-velocity as usual.

(b) Use this to show that

$$\nabla_\mu T^{\mu\nu}_{pp} = 0$$

implies that the $y^\alpha(\tau)$ obeys the geodesic equation. Cool, huh?
(Hint: plug part a into part b, and convert the $x-$derivative to a y-derivative. Then integrate by parts.)

(c) We saw in class that the action

$$S_{pp} = m \int d\tau = m \int d\tau \left[-g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right]^{1/2}$$

gave the geodesic equation as an equation of motion for $x^\mu(\tau)$.
As discussed in class, we can obtain the energy-momentum tensor by varying the action with respect to the metric. Show that

$$\frac{-2}{\sqrt{-g}} \frac{\delta S_{pp}}{\delta g^{\mu\nu}} = T_{\mu\nu,pp},$$

with $T_{\mu\nu,pp}$ as given in Eq. 7

Solution:

(a) We have

$$\begin{aligned} \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} T^{\mu\nu}) + \Gamma^\nu_{\alpha\beta} T^{\alpha\beta} &= \\ \partial_\mu T^{\mu\nu} + T^{\mu\nu} \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}) + \Gamma^\nu_{\alpha\beta} T^{\alpha\beta} &= \\ \partial_\mu T^{\mu\nu} + T^{\mu\nu} \Gamma^\alpha_{\mu\alpha} + \Gamma^\nu_{\alpha\beta} T^{\alpha\beta} &= \nabla_\mu T^{\mu\nu} \end{aligned} \tag{8}$$

where the third line follows from Carroll Eq. 3.33.

(b) First, since g depends just on x and not y, we can pull then $1/\sqrt{-g}$ out of $T^{\mu\nu}_{pp}$. Then, using the result of part (a) and noting that the $\partial_\mu = \partial/\partial x^\mu$ does not act on U^ν (which is only a function of y), we get:

$$0 = \int U^\mu U^\nu \partial_\mu \delta^4(x - y(\tau)) + \Gamma^\nu_{\sigma\mu} \int U^\mu U^\sigma \delta^4(x - y(\tau)) d\tau$$

Since the δ depends only on the difference between x and y, we can replace the $\partial/\partial x^\mu$ with $-\partial/\partial y^\mu$. We then have

$$U^\mu \frac{\partial}{\partial x^\mu} \delta^4(x - y(\tau)) = -U^\mu \frac{\partial}{\partial y^\mu} \delta^4(x - y(\tau)) = -\frac{d}{d\tau} \delta^4(x - y(\tau)),$$

since $U^\mu = dy^\mu/d\tau$. This gives

$$0 = -\int U^\nu \frac{d}{d\tau} \delta^4(x - y(\tau)) + \Gamma^\nu_{\sigma\mu} \int U^\mu U^\sigma \delta^4(x - y(\tau)) d\tau.$$

Doing an integration by parts on the first term, we then get

$$\int \left[\frac{d}{d\tau} U^\nu + \Gamma^\nu_{\sigma\mu} \int U^\mu U^\sigma \right] \delta^4(x - y(\tau)) d\tau = 0$$

For this integral to vanish we must have the expression in brackets vanish along the particle's trajectory (though only there), so we regain the geodesic equation.

(c) I failed to assign this part of the problem, but here is the solution anyway, since it is nice.

We have

$$\begin{aligned} \frac{\delta S_{pp}}{\delta g^{\mu\nu}} &= \delta \left[m \int d\tau \left\{ g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\mu}{d\tau} \right\}^{1/2} \right] \\ &= \frac{m}{2} \int d\tau \left\{ g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\mu}{d\tau} \right\}^{-1/2} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \delta g_{\alpha\beta} \\ &= \frac{m}{2} \int d\tau \, \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} g_{\alpha\gamma} g_{\beta\delta} \delta g^{\beta\delta} \end{aligned} \tag{9}$$

Now we can sneak in a δ function:

$$\begin{aligned} \frac{-2}{\sqrt{-g(y)}} \delta S_{pp}(y) &= \frac{-m}{\sqrt{-g(y)}} \int \sqrt{-g}\, d^4x \int d\tau\, \delta^4(x - y) \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} g_{\alpha\gamma} g_{\beta\delta} \delta g^{\beta\delta} \\ &= \int \sqrt{-g(x)}\, d^4x \frac{-m}{\sqrt{-g(x)}} \int d\tau\, \delta^4(x - y) \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} g_{\alpha\gamma} g_{\beta\delta} \delta g^{\beta\delta} \\ &= \int d\tau \int d^4x\, \delta^4(x - y) \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} g_{\alpha\gamma} g_{\beta\delta} \delta g^{\beta\delta}. \end{aligned} \tag{10}$$

Then using the definition of the functional derivative,

$$\delta J \equiv \int dx \frac{\delta J}{\delta f(x)} \delta f(x),$$

we find

$$\frac{-2}{\sqrt{-g(y)}}\frac{\delta S_{pp}(y)}{\delta g^{\gamma\delta}} = \int d\tau\, \delta^4(x-y)\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}g_{\alpha\gamma}g_{\beta\delta} = T_{\gamma\delta,pp}.$$