Physics/Astronomy 226, Problem set 8, Due 3/10

1. There is extremely strong astrophysical evidence that black holes of mass $10^6 - 10^8\, M_\odot$ reside in the centers of galaxies, and our own galaxy probably hosts a (probably Kerr) black hole of $\sim 10^6\, M_\odot$. Assume $a = 0$, (and $Q = 0$), and $M = 10^6\, M_\odot$ for present purposes.

   (a) Find the radius (in A.U.) of the horizon of our galaxy's black hole.

   (b) The Next-next-next-next Generation Space Telescope (NNNNGST) is observing the black hole from the innermost stable circular orbit. NNNNGST sends a packet of observational data along a radial null geodesic to a data analysis lab (at fixed $r \gg GM, \theta$ and ϕ) each time it orbits the black hole. What is the interval between such transmissions according to NNNNGST's internal clock? How long must astronomers in the lab wait between packets?

2. Consider Einstein's Equations in vacuum, but with a cosmological constant: $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$. Solve for the most general spherically symmetric metric in coordinates (t, r, θ, ϕ) such that the metric reduces to the Schwarzschild one when $\Lambda = 0$. (Hint: write the EEs in terms of $R_{\mu\nu}$ rather than $G_{\mu\nu}$ by moving Λ to the r.h.s. The solution then closely follows Carroll's Sec. 5.1-5.2.)

3. Consider once again the 'wormhole' metric:

$$ds^2 = -dt^2 + dr^2 + (b^2 + r^2)d\Omega^2,$$

   where $-\infty < t < \infty$, $-\infty < r < \infty$. Previously, you have computed $T_{\mu\nu}$ and other tensors for this metric, and the undergrads in the class have computed embedding diagrams for it.

   (a) Define a conserved energy E and angular-momentum L in terms of r, ϕ, and their derivatives.

   (b) Show that an observer who falls freely and radially in this spacetime moves along the worldline $r = vt$, θ=const., ϕ=const., where v =const.< 1

   (c) Derive an equation like Carroll's 5.65-5.67, i.e. a 1-D particle in an effective potential.

   (d) Analyze the orbits in this geometry, in the same manner we did for the Schwarzschild metric.