Physics/Astronomy 226, Problem set 9, Due *3/19*

1. Letting $u = (t - z)/\sqrt{2}$ and $v = (t + z)/\sqrt{2}$ transforms the usual Minkowski metric $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$ into $ds^2 = -(dudv + dvdu) + dx^2 + dy^2$. This suggests considering the metric ansatz:

$$ds^2 = -(dudv + dvdu) + a^2(u)dx^2 + b^2(u)dy^2,$$

where a and b are unspecified functions of u. For appropriate a and b this is an exact gravitational place wave propagating in the $\hat{z}$ direction.

   (a) Calculate the Christoffel symbols and Riemann tensor for this metric (You'll save a lot of time by reading them off of the geodesic eqns. you get by varying the action.)

   (b) Use the vacuum Einstein Equations $R_{\mu\nu} = 0$ to get equations for a and b.

   (c) Show that an exact solution can be found, in which both a and b are determined (up to integration constants) by an arbitrary function $f(u)$.

2. The area of the outer horizon of a Kerr black hole is given by

$$A = 4\pi(r_+^2 + a^2),$$

where

$$r_+ = GM + \sqrt{G^2M^2 - a^2}.$$

Hawking has proven that if there are no naked singularities ("cosmic censorship") and the WEC holds, then in an asymptotically flat spacetime, the sum of the areas of the future event horizons of a set of black holes never decreases.

Assuming that the WEC and cosmic censorship hold, consider the collision of two Kerr black holes of equal mass M_k, and angular momentum of equal magnitude but opposite direction (i.e. both have spin parameter a, but rotate in opposite senses.)

   (a) What is the minimal mass M_s of a Schwarzschild black hole that forms?

   (b) If $|a| \approx M$ for the initial Kerr holes, what fraction of the original energy $2M_k$ can be lost? Where does this energy go?

3. I feel a bit bad that I have blithely skipped over some of the key historical tests of GR that led to us thinking it is actually true. To assuage my guilt:

   (a) Consider the weak-field Einstein equations in Lorenz gauge in terms of the trace-reverse $\bar{h}_{\mu\nu}$ of the perturbation $h_{\mu\nu}$:

$$\Box\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}.$$

   Re-write this into an equation with $\Box h_{\mu\nu}$ on the l.h.s. and the energy-momentum tensor $T_{\mu\nu}$ and its trace T on the r.h.s.

(b) Let $T_{\mu\nu}$ be that for dust of rest-frame density ρ, and go to the rest frame of the dust. Consider the field $h_{\mu\nu}$ to be slowly changing, i.e. $\partial_i h_{\mu\nu} \gg \partial_0 h_{\mu\nu}$. Letting Φ be the Newtonian potential ($\nabla^2\Phi = 4\pi G\rho$), write the metric solving Einstein's equations to first order in Φ. Check your result with Carroll's Eq. 7.59

(c) Put your GR machinery to work: calculate the Christoffel symbols (to first order in Φ), and write the geodesic equation for the momentum $p^\mu = dx^\mu/d\lambda$ of a photon.

(d) Imagine this photon moving in the $\hat{z}$ direction and passing by a spherically symmetric mass M, so that at closest approach, the separation between the mass and the photon is a distance b in the $\hat{x}$ direction. Assume from the start that the deflection angle θ is very small. Calculate this angle by estimating it to be $\theta \simeq (\Delta p^x)/p^z$, where Δp^x is calculated by integrating the x-component of the geodesic equation along the *unperturbed* (x, y, p^α=const.) path.

(e) Put in appropriate factors of c, and estimate this deflection angle for $b = 2R_\odot$ and $M = M_\odot$. Do the same for $M = M_{\text{jupiter}}$ and $b = 1.1R_{\text{jupiter}}$.