Physics/Astronomy 226, Problem set 8, Due 3/10
Solutions

1. There is extremely strong astrophysical evidence that black holes of mass $10^6 - 10^8\, M_\odot$ reside in the centers of galaxies, and our own galaxy probably hosts a (probably Kerr) black hole of $\sim 10^6\, M_\odot$. Assume $a = 0$, (and $Q = 0$), and $M = 10^6\, M_\odot$ for present purposes.

   (a) Find the radius (in A.U.) of the horizon of our galaxy's black hole.

   (b) The Next-next-next-next Generation Space Telescope (NNNNGST) is observing the black hole from the innermost stable circular orbit. NNNNGST sends a packet of observational data along a radial null geodesic to a data analysis lab (at fixed $r \gg GM, \theta$ and ϕ) each time it orbits the black hole. What is the interval between such transmissions according to NNNNGST's internal clock? How long must astronomers in the lab wait between packets?

   Solution:

   (a) The radius of the Schwarzschild event horizon is $R = \frac{2GM}{c^2}$ where in this case $M = 10^6 M_\odot$
   From equation (5.96), $\frac{GM_\odot}{c^2} = 1.48 \cdot 10^3 m$, which yields $R = 2.96 \cdot 10^9 m$.
   To convert to AU, note $1AU = 1.495 \cdot 10^{11} m$. Then $R = .0198AU$.

   (b) The radius of the innermost stable circular orbit is $r_c = 6\frac{GM}{c^2}$ or $r_c = 8.88 \cdot 10^9 m$.
   The time observed between transmissions by the orbiting telescope, τ, is found using $L = {r_c}^2 \frac{d\phi}{d\tau}$. Integrating, $\int d\tau = \frac{{r_c}^2}{L} \int_0^{2\pi} d\phi$ or $\tau = \frac{2\pi {r_c}^2}{L}$.
   For this orbit, $L = \frac{\sqrt{12}GM}{c}$, so $\tau = \frac{2\pi(8.88 \cdot 10^9 m)^2}{\sqrt{12}(3 \cdot 10^8 m/s)(1.48 \cdot 10^9 m)} = 322s$
   The time observed in the lab will be dilated due to gravitational redshift. The shift $\frac{dt}{d\tau}$ appears in the equation

$$E = (1 - \frac{2GM}{r})\frac{dt}{d\tau}$$

   Using $L = \sqrt{12}GM$ and $r = 6GM$, the energy can be calculated from

$$\frac{E^2}{2} = V(r)$$

   This equation is obtained from eqn(5.65) of Carroll, with $\frac{dr}{d\lambda}$ set to 0. Substituting L and r into $V(r) = \frac{1}{2} - \frac{GM}{r} + \frac{L^2}{2r^2}\frac{GML^2}{r^3}$ yields

$$\frac{E^2}{2} = \frac{4}{9}$$

   or

$$\frac{2\sqrt{2}}{3} = (1 - \frac{2GM}{r})\frac{dt}{d\tau}$$

   The shift is then $\sqrt{2} = 1.41$, giving $\Delta t_{lab} = 1.41 \cdot 322s = 455s$.

2. Consider Einstein's Equations in vacuum, but with a cosmological constant: $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$. Solve for the most general spherically symmetric metric in coordinates (t, r, θ, ϕ) such that the metric reduces to the Schwarzschild one when $\Lambda = 0$. (Hint: write the EEs in terms of $R_{\mu\nu}$ rather than $G_{\mu\nu}$ by moving Λ to the r.h.s. The solution then closely follows Carroll's Sec. 5.1-5.2.)

Solution:

Using Einstein's Equations with a Cosmological constant, we have:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0$$

contracting with $g^{\mu\nu}$ we get $R = 4\Lambda$. Plugging this back into Einstein's equations, they now read:

$$\Lambda g_{\mu\nu} = R_{\mu\nu} \quad (i)$$

Using this equation we can constrain the most general spherically symmetric solution to the metric in a vaccum. This is given in Carroll by the equation:

$$ds^2 = -e^{2\alpha}dt^2 + e^{2\beta}dr^2 + r^2 d\Omega^2 \quad (ii)$$

Here α and β are functions of r and t. Using Birkhoff's Theorem we can use Einstein's equations to constrain β and α placing all the time dependence of the metric on α, then we can re-define the time coordinate to take out all explicit time dependence from α. The algebra involved here is similar to the procedure I will employ below in another similar step in this problem. In the end we will get a metric that looks just like equation (ii) above but with α and β only dependent on r in our new coordinate system. From this new metric it is possible to write all of the components of the Reimann Curvature tensor(and thus the Ricci tensor). The components that will be employed to solve this problem are $R_{\theta\theta}$, R_{tt}, and R_{rr}. From Carroll we have:

$$R_{tt} = e^{2(\alpha-\beta)}(\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha)$$

$$R_{rr} = -\partial_r^2\alpha - (\partial_r\alpha)^2 + \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\beta$$

$$R_{\theta\theta} = e^{-2\beta}(r(\partial_r\beta - \partial_r\alpha) - 1) + 1$$

From the proper time interval above we can see that $g_{tt} = -e^{2\alpha}$, $g_{rr} = e^{2\beta}$, and $g_{\theta\theta} = r^2$. Plugging these values into equation (i) above we will get three equations back, one for the tt component, one for the rr component, and one for the $\theta\theta$ component. Dividing each side of the tt equation by $e^{2\alpha}$ and then adding this equation to the rr equation multiplied by $e^{-2\beta}$, many terms cancel and we are left with the equation:

$$\frac{2}{r}(\partial_r\alpha + \partial_r\beta) = 0$$

Pulling out the ∂_r we can see that $\alpha - \beta = c$ where c is a constant. If we re-label the time coordinate such that $t \rightarrow te^{-c}$ then we can absorb e^{-c} into the new scaling of our time and have $\alpha = -\beta$ if we maintain the same form of the metric as given above.

With that we can now solve for α and then have β and thus have the whole metric. Now we will use the $\theta\theta$ component of equation (i) to solve for α:

$$R_{\theta\theta} = \Lambda g_{\theta\theta}$$

$$e^{-2\beta}(r(\partial_r \beta - \partial_r \alpha) - 1) + 1 = \Lambda r^2$$

use $\alpha = -\beta$ and write the equation for α:

$$e^{2\alpha}(2r(\partial_r \alpha) + 1) = -\Lambda r^2 + 1$$

notice that doing the product rule backwards says:

$$\partial_r(re^{2\alpha}) = -\Lambda r^2 + 1$$

We can integrate this equation and find:

$$re^{2\alpha} = \frac{-\Lambda r^3}{3} + r + C$$

Here C is just an arbitrary integration constant, it will become obvious what it is in a moment. Solving for $-e^{2\alpha}$ gives g_{tt} by definition. And since $\alpha = -\beta$ $g_{rr} = e^{-2\alpha}$. Also, we can note that the integration constant must $= -2GM$ when we compare to the Schwarzchild metric in the limit that $\Lambda \to 0$. In the end we find:

$$g_{tt} = (1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3})$$

$$g_{rr} = \frac{1}{1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}}$$

$$g_{\theta\theta} = r^2$$

$$g_{\phi\phi} = r^2 \sin\theta$$

3. Consider once again the 'wormhole' metric:

$$ds^2 = -dt^2 + dr^2 + (b^2 + r^2)d\Omega^2,$$

where $-\infty < t < \infty$, $-\infty < r < \infty$. Previously, you have computed $T_{\mu\nu}$ and other tensors for this metric, and the undergrads in the class have computed embedding diagrams for it.

(a) Define a conserved energy E and angular-momentum L in terms of r, ϕ, and their derivatives.

(b) Show that an observer who falls freely and radially in this spacetime moves along the worldline $r = vt$, θ=const., ϕ=const., where v =const.< 1

(c) Derive an equation like Carroll's 5.65-5.67, i.e. a 1-D particle in an effective potential.

(d) Analyze the orbits in this geometry, in the same manner we did for the Schwarzschild metric.

Solution:

Don't have one yet.