Physics/Astronomy 226, Problem set 7, Due 3/3

Reading: Carroll, Ch. 4 still

1. Show that for a Killing vector K^ρ, and with no torsion (as usual),

$$\nabla_\mu \nabla_\sigma K_\rho = R^\nu{}_{\mu\sigma\rho} K_\nu \quad \text{and from this, } \nabla_\mu \nabla_\sigma K^\mu = R_{\sigma\nu} K^\nu.$$

(Hint: use the identity for $R^\nu{}_{\mu\sigma\rho}$ in which the sum of three permutations vanishes).

Use this, the Bianchi identity, and Killing's Equation to show:

$$K^\lambda \nabla_\lambda R = 0.$$

2. Let S be the set of points in 2-D Minkowski space (with Cartesian coordinates x, t) for which $|x| \le 1$ and $t = 0$. Let $J^\pm$ and $I^\pm$ be the causal and chronological past/future, respectively. Let $D^\pm$ and $H^\pm$ respectively be the the past/future domain of dependence and Cauchy horizon. You may draw your answers as long as things are carefully labeled and unambiguous; otherwise specify the sets mathematically.

   (a) What is $J^-(S)$?
   (b) For what points p is $S \subset J^+(p)$?
   (c) What is $J^+(S) - I^+(S)$? Is this set achronal?
   (d) What is $D^+(S)$? What is $H^+(S)$?
   (e) What is $H^+(\partial I^+(S))$? How about $H^-(\partial I^+(S))$, where ∂ denotes the boundary of the set? (Don't think of a boundary at infinity, just the 'lower' boundary.)

3. Draw the conformal diagram for Minkoskwi space (it will be helpful to read Appendix H of Carroll). Now draw and label examples of:

   (a) The path ("worldline") of a particle following a timelike geodesic.
   (b) The worldline of a particle following a null geodesic.
   (c) A Cauchy surface for the spacetime.
   (d) The worldline of a particle that follows a timelike geodesic until some time, then undergoes constant acceleration forever.
   (e) $J^-(S)$, where S is the worldline of part (d).

4. Consider the metric

$$ds^2 = -dt^2 + dr^2 + (b^2 + r^2)d\Omega^2.$$

This describes a wormhole, which we will investigate later. For now, calculate $T_{\mu\nu}$ assuming this satisfies Einstein's equations. Which energy conditions does this spacetime violate? (Time-saving hint: feel free to use Mathematica for this. There are GR packages for Mathematica, but I suspect will take some time investment to get working. For a quick solution, there is a nice notebook you can modify for this purpose at: http://wps.aw.com/aw_hartle_gravity_1/0,6533,512494-,00.html. Note, however, that he sets $x^3 = t$, not $x^0 = t$, which you have to watch out for.)