Physics/Astronomy 226, Problem set 7, Due 3/3
Solutions

Reading: Carroll, Ch. 4 still

1. Show that for a Killing vector K^ρ, and with no torsion (as usual),

$$\nabla_\mu \nabla_\sigma K_\rho = R^\nu{}_{\mu\sigma\rho} K_\nu \quad \text{and from this, } \nabla_\mu \nabla_\sigma K^\mu = R_{\sigma\nu} K^\nu.$$

(Hint: use the identity for $R^\nu{}_{\mu\sigma\rho}$ in which the sum of three permutations vanishes). Use this, the Bianchi identity, and Killing's Equation to show:

$$K^\lambda \nabla_\lambda R = 0.$$

Solution:

First, start with the formula from Carroll (equation (3.112))

$$[\nabla_\mu, \nabla_\sigma] K^\rho = R^\rho{}_{\nu\mu\sigma} K^\nu$$

Lowering the indices gives

$$[\nabla_\mu, \nabla_\sigma] K_\rho = R_{\rho\nu\mu\sigma} K^\nu = R_{\nu\rho\sigma\mu} K^\nu = R^\nu{}_{\rho\sigma\mu} K_\nu \tag{1}$$

Now, using (1) and the symmetry properties of the curvature tensor and Killing vector, i.e.

$$\begin{aligned} R^\nu{}_{[\mu\sigma\rho]} = 0 \quad &\text{or} \quad R^\nu{}_{\mu\sigma\rho} + R^\nu{}_{\sigma\rho\mu} + R^\nu{}_{\rho\mu\sigma} = 0 \\ \nabla_{(\mu} K_{\nu)} = 0 \quad &\text{or} \quad \nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 \end{aligned} \tag{2}$$

we have

$$\begin{aligned} \nabla_\mu \nabla_\sigma K_\rho &= \nabla_\sigma \nabla_\mu K_\rho + R^\nu{}_{\rho\sigma\mu} K_\nu \\ &= \nabla_\sigma \nabla_\mu K_\rho - (R^\nu{}_{\sigma\mu\rho} + R^\nu{}_{\mu\rho\sigma}) K_\nu \\ &= R^\nu{}_{\mu\sigma\rho} K_\nu + (\nabla_\sigma \nabla_\mu K_\rho + R^\nu{}_{\sigma\rho\mu} K_\nu) \end{aligned}$$

where (using (1))

$$\begin{aligned} \nabla_\sigma \nabla_\mu K_\rho - R^\nu{}_{\sigma\mu\rho} K_\nu &= \nabla_\sigma \nabla_\mu K_\rho + [\nabla_\mu, \nabla_\rho] K_\sigma \\ &= \nabla_\sigma \nabla_\mu K_\rho + \nabla_\mu \nabla_\rho K_\sigma + \nabla_\rho \nabla_\sigma K_\mu \quad \text{(using K.E.)} \\ &= \frac{1}{2} \left\{ \begin{array}{l} \nabla_\sigma \nabla_\mu K_\rho + \nabla_\mu \nabla_\rho K_\sigma + \nabla_\rho \nabla_\sigma K_\mu \\ -\nabla_\sigma \nabla_\rho K_\mu - \nabla_\mu \nabla_\sigma K_\rho - \nabla_\rho \nabla_\mu K_\sigma \end{array} \right\} \text{(using K.E.)} \\ &= \frac{1}{2} ([\nabla_\sigma, \nabla_\mu] K_\rho + [\nabla_\mu, \nabla_\rho] K_\sigma + [\nabla_\rho, \nabla_\sigma] K_\mu) \\ &= \frac{1}{2} (R^\nu{}_{\rho\mu\sigma} + R^\nu{}_{\sigma\rho\mu} + R^\nu{}_{\mu\sigma\rho}) K_\nu \\ &= 0 \end{aligned}$$

So,

$$\nabla_\mu \nabla_\sigma K_\rho = R^\nu{}_{\mu\sigma\rho} K_\nu \tag{3}$$

and contracting this expression gives

$$\begin{aligned}
\nabla^\mu \nabla_\sigma K_\mu &= g^{\rho\mu} \nabla_\mu \nabla_\sigma K_\rho = g^{\rho\mu} R^\nu{}_{\mu\sigma\rho} K_\nu \\
&= g^{\rho\mu} R_{\nu\mu\sigma\rho} K^\nu = R^\rho{}_{\nu\rho\sigma} K^\nu \\
&= R_{\sigma\nu} K^\nu
\end{aligned}$$

Therefore,

$$\nabla_\mu \nabla_\sigma K^\mu = R_{\sigma\nu} K^\nu \tag{4}$$

Using (4) with the Bianchi identity and the symmetry of the Ricci tensor,

$$\left.\begin{aligned}
&\nabla^\mu R_{\lambda\mu} = \tfrac{1}{2} \nabla_\lambda R \\
&R_{\lambda\mu} = R_{\mu\lambda} = R_{(\lambda\mu)}
\end{aligned}\right\} \tag{5}$$

yields

$$\begin{aligned}
K^\lambda \nabla_\lambda R &= 2K^\lambda \nabla^\alpha R_{\lambda\alpha} \\
&= 2\nabla^\alpha (K^\lambda R_{\lambda\alpha}) - R_{\lambda\alpha} \nabla^\lambda K^\alpha \\
&= 2\nabla^\alpha K^\lambda R_{\lambda\alpha} \\
&= 2\nabla^\alpha \nabla_\mu \nabla_\alpha K^\mu
\end{aligned}$$

But this reduces to,

$$\begin{aligned}
2\nabla^\alpha \nabla_\mu \nabla_\alpha K^\mu &= [\nabla_\alpha, \nabla_\mu] \nabla^\alpha K^\mu + 2\nabla_{(\alpha} \nabla_{\mu)} \nabla^\alpha K^\mu \\
&= [\nabla_\alpha, \nabla_\mu] \nabla^\alpha K^\mu + 2\nabla_{(\alpha} \nabla_{\mu)} \nabla^{(\alpha} K^{\mu)} \\
&= R^\alpha{}_{\beta\alpha\mu} \nabla^\beta K^\mu + R^\mu{}_{\beta\alpha\mu} \nabla^\alpha K^\beta \\
&= R_{\beta\mu} \nabla^\beta K^\mu - R_{\beta\alpha} \nabla^\alpha K^\beta \\
&= 0
\end{aligned}$$

and hence,

$$K^\lambda \nabla_\lambda R = 0. \tag{6}$$

Whew!

2. Let S be the set of points in 2-D Minkowski space (with Cartesian coordinates x, t) for which $|x| \leq 1$ and $t = 0$. Let $J^\pm$ and $I^\pm$ be the causal and chronological past/future, respectively. Let $D^\pm$ and $H^\pm$ respectively be the the past/future domain of dependence and Cauchy horizon. You may draw your answers as long as things are carefully labeled and unambiguous; otherwise specify the sets mathematically.

(a) What is $J^-(S)$?

(b) For what points p is $S \subset J^+(p)$?

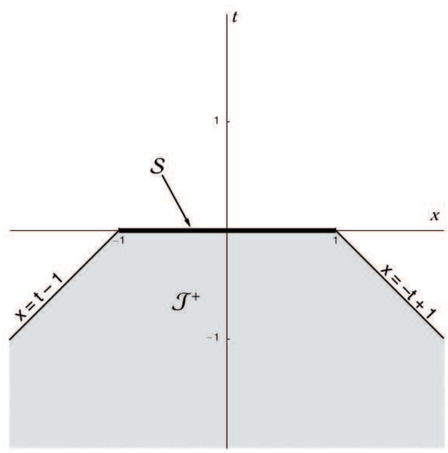

FIG. 1: The region $J^{-}(S)$ includes the boundary lines.

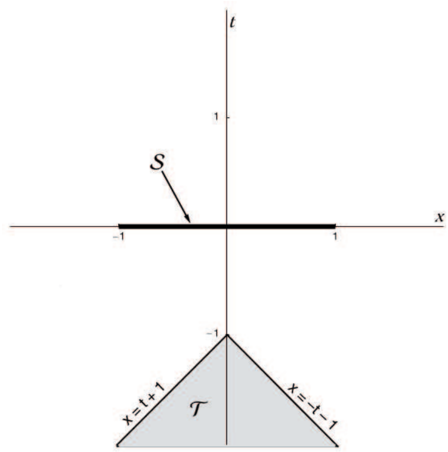

FIG. 2: T is the set of points p for which $S \subset J^{+}(p)$; this region includes the boundaries.

(c) What is $J^{+}(S) - I^{+}(S)$? Is this set achronal?

(d) What is $D^{+}(S)$? What is $H^{+}(S)$?

(e) What is $H^{+}(\partial I^{+}(S))$? How about $H^{-}(\partial I^{+}(S))$, where ∂ denotes the boundary of the set? (Don't think of a boundary at infinity, just the 'lower' boundary.)

Solution:

3. Draw the conformal diagram for Minkoskwi space (it will be helpful to read Appendix H of Carroll). Now draw and label examples of:

(a) The path ("worldline") of a particle following a timelike geodesic.

(b) The worldline of a particle following a null geodesic.

(c) A Cauchy surface for the spacetime.

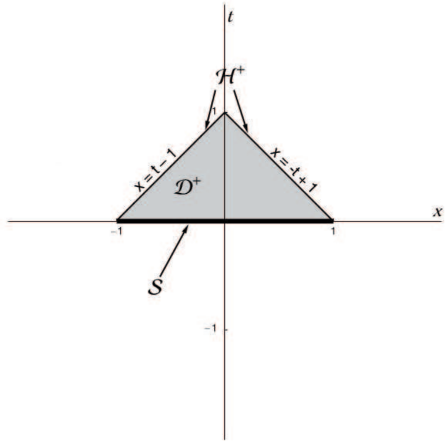

FIG. 3: The region $J^+(S) - I^+(S)$ is just the boundary lines of figure 1. This set is achronal–it only contains null curves.

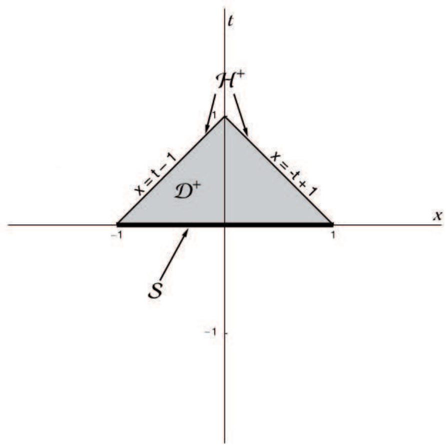

FIG. 4: The set $D^+(S)$ includes the boundary lines. $H^+(S)$ is just the boundaries of $D^+(S)$.

(d) The worldline of a particle that follows a timelike geodesic until some time, then undergoes constant acceleration forever.

(e) $J^-(S)$, where S is the worldline of part (d).

Solution:

4. Consider the metric

$$ds^2 = -dt^2 + dr^2 + (b^2 + r^2)d\Omega^2.$$

This describes a wormhole, which we will investigate later. For now, calculate $T_{\mu\nu}$ assuming this satisfies Einstein's equations. Which energy conditions does this spacetime violate? (Time-saving hint: feel free to use Mathematica for this. There are GR packages for Mathematica, but I suspect will take some time investment to get working. For a quick solution, there is a nice notebook you can modify for this purpose at:

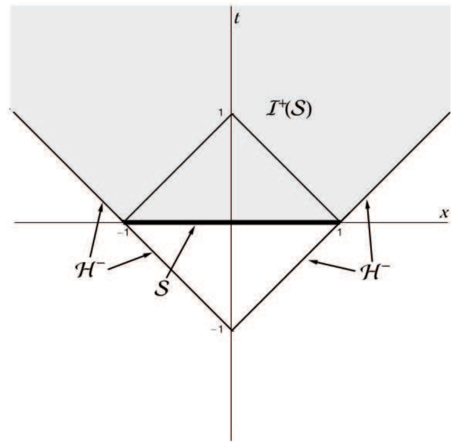

FIG. 5: The set $H^+(I^+(S))$ is the boundary of the x plane at $t = \infty$. The set $H^-(I^+(S))$ is just the two boundary lines of $I^+(S)$ extended until they almost intersect. Note: technically $D^{\pm}(S)$ is only defined for *achronal* S

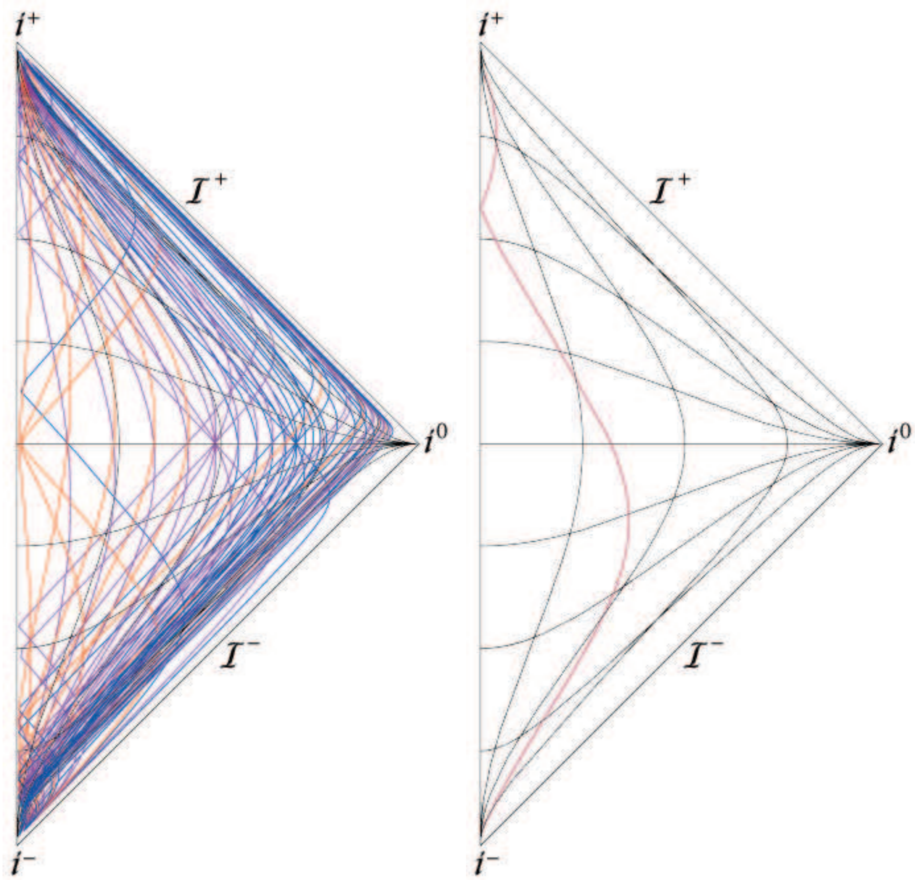

FIG. 6: Paths of timelike geodesics. On the left is a whole family of geodesics, parameterized by $r = |\alpha(t - \gamma) + \beta|$; on the right is a single geodesic $r = |0.4t - 0.6|$.

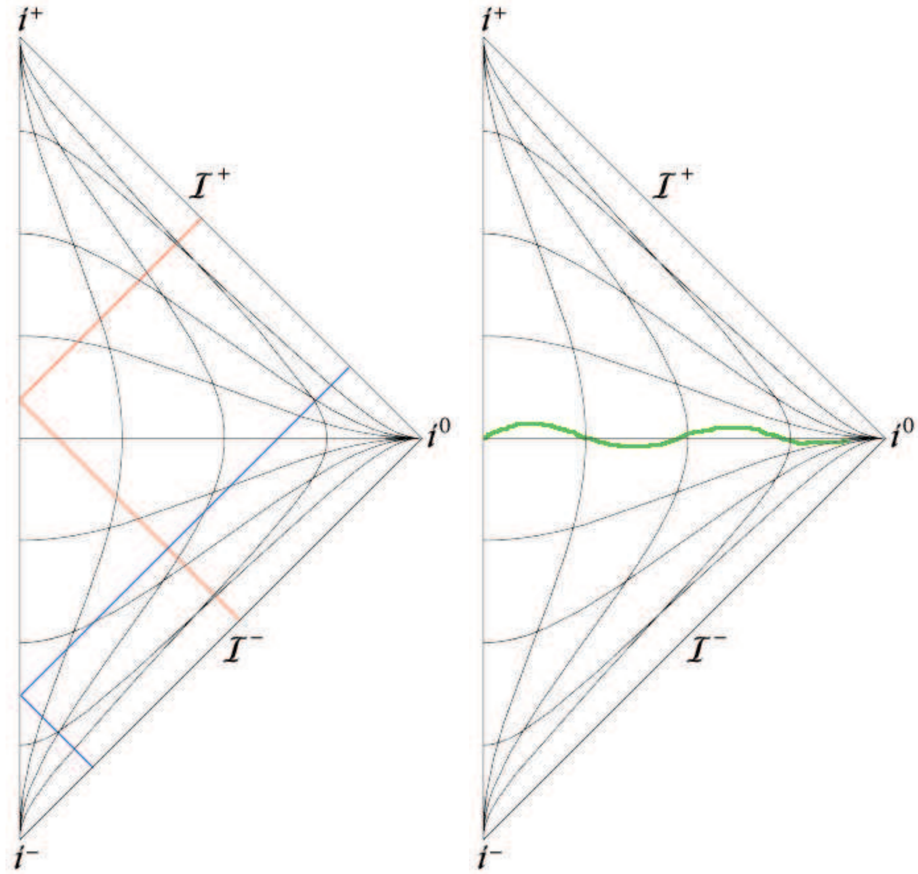

FIG. 7: On the left are a couple null radial geodesics. On the right is a Cauchy surface for the spacetime.

http://wps.aw.com/aw_hartle_gravity_1/0,6533,512494-,00.html. Note, however, that he sets $x^3 = t$, not $x^0 = t$, which you have to watch out for.)

Solution: Modifying the notebook as suggested spits out:

$$G_{rr} = G_{tt} = -G_{\theta\theta} = -G_{\phi\phi}/\sin^2\theta = -\frac{b^2}{(b^2+r^2)^2}.$$

Now, if we let $t^\mu = (1,0,0,0)$, it's clear that

$$T_{\mu\nu}t^\mu t^\nu = \frac{1}{8\pi G}G_{00} = -\frac{1}{8\pi G}\frac{b^2}{(b^2+r^2)^2}$$

is negative. Thus the WEC (at least) is violated.

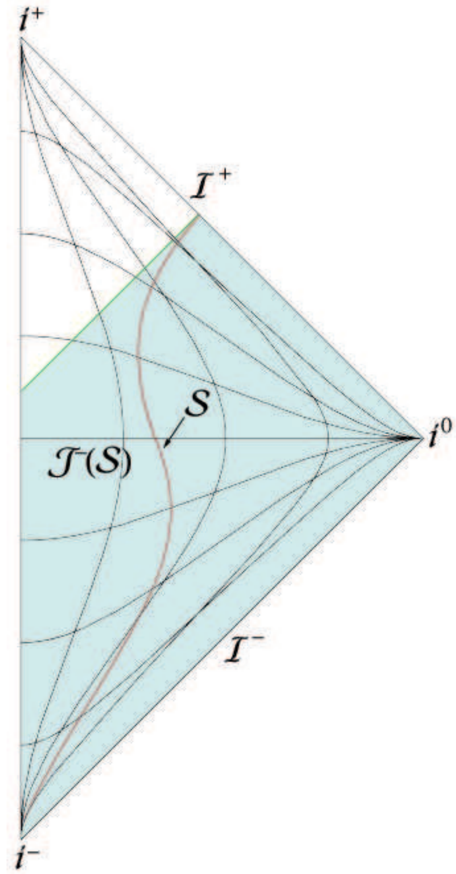

FIG. 8: S is the worldline of a particle that follows a geodesic for $t < 0$ and is accelerating for $t > 0$. Notice it asymptotically approaches the null geodesic (green). The set $J^-(S)$ is the (blue) shaded region, not including the null geodesic boundary.