Phys 101B — HW Solutions

1. $$H = \sum_{i=1}^{Z} \left(\frac{-\hbar^2}{2m} \vec{\nabla}_i^2 + V(r_i) \right) \qquad \text{where } \vec{\nabla}_i^2 = \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + \frac{\partial^2}{\partial z_i^2}$$

The joint wavefunction satisfies $H\Psi = E\Psi$:

$$\sum_{i=1}^{Z} \left(-\frac{\hbar^2}{2m} \vec{\nabla}_i^2 + V(r_i) \right) \Psi = E\Psi$$

pluggin in $\Psi(\vec{x}_1, \dots \vec{x}_Z) = \psi_1(\vec{x}_1)\psi_2(\vec{x}_2) \cdots \psi_Z(\vec{x}_Z)$:

$$\sum_{i=1}^{Z} \psi_1(\vec{x}_1) \cdots \psi_{i-1}(x_{i-1}) \left[\left(\frac{-\hbar^2}{2m} \vec{\nabla}_i + V_i \right) \psi_i(\vec{x}_i) \right] \psi_{i+1}(\vec{x}_{i+1}) \cdots \psi_Z(\vec{x}_Z) = E\, \psi_1(\vec{x}_1) \cdots \psi_Z(\vec{x}_Z)$$

Dividing both sides by Ψ isolates each $\vec{x}_i$ variable:

$$\sum_{i=1}^{Z} \frac{1}{\psi_i(\vec{x}_i)} \left(\frac{-\hbar^2}{2m} \vec{\nabla}_i + V(r_i) \right) \psi_i(\vec{x}_i) = E$$

Now each term depends only on $\vec{x}_i$ so they must all be constants by separation of variables. Calling the i^{th} term E_i, then $\sum_{i=1}^{Z} E_i = E$ and:

$$\frac{1}{\psi_i(\vec{x}_i)} \left(-\frac{\hbar^2}{2m} \vec{\nabla}_i + V(r_i) \right) \psi_i(\vec{x}_i) = E_i \text{ , or:}$$

$$\left[\frac{-\hbar^2}{2m} \vec{\nabla}_i + V(r_i) \right] \psi_i(\vec{x}_i) = E_i \psi_i(\vec{x}_i)$$

2. $$S_y \psi_{-y} = \frac{\hbar}{2} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} -1 \\ -i \end{pmatrix} = -\frac{\hbar}{2} \psi_{-y} \rightarrow \text{eigenvalue } -\frac{\hbar}{2}$$

$$S_y \psi_{+y} = \frac{\hbar}{2} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{\hbar}{2} \psi_{+y} \rightarrow \text{eigenvalue of } \frac{\hbar}{2}$$

Since $H = \left(\frac{eB}{M}\right) S_y$, the energy eigenvalues are just $\left(\frac{eB}{M}\right)$ times those of S_y: $E_- = \frac{-eB\hbar}{2M}$, $E_+ = \frac{eB\hbar}{2M}$

Since ψ_{-y} & ψ_y are energy eigenstates they have the time dependence:

$$\psi_{-y}(t) = \psi_{-y}(0) e^{-iE_- t/\hbar}, \quad \psi_{+y}(t) = \psi_{+y}(0) e^{-iE_+ t/\hbar}$$

By linearity, $\psi(t)$ is then: $\psi(t) = \frac{1}{2}\left(\psi_{-y}(t) + \psi_{+y}(t)\right)$

$$\psi(t) = \frac{1}{2} \begin{pmatrix} e^{i\left(\frac{eB}{2M}\right)t} + e^{-i\left(\frac{eB}{2M}\right)t} \\ i\left(e^{i\left(\frac{eB}{2M}\right)t} - e^{-i\left(\frac{eB}{2M}\right)t}\right) \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{eBt}{2M}\right) \\ -\sin\left(\frac{eBt}{2M}\right) \end{pmatrix}$$

The probability of spin-down $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ at time t is then:

$$\sin^2\left(\frac{eBt}{2M}\right)$$