Physics 101B. Modern Physics. Professor Dine

Winter, 2005. Some Very Modern Physics: Spin 1/2 and a Connection to Neutrinos

For spin, we can think of a "wave function", but with just two possible states, corresponding to z component of spin $\pm 1/2\hbar$

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1}$$

The notation on the left is due to Dirac, he represented a state of a system, abstractly, by a "bra", $|\ \rangle$. Then the spin operator is just a very simple matrix:

$$S_z = \hbar/2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2}$$

The states are eigenstates of this matrix, with eigenvalues $\pm\frac{\hbar}{2}$.

The energy is:

$$H = \frac{g S_z B_z}{2m_e} \tag{3}$$

Now suppose we start out by preparing the system in the state of spin up. Then we pass it through a magnetic field *in the x direction.* Now the Hamiltonian is different. I claim it is:

$$H = \frac{g S_x B_x}{2m_e} \tag{4}$$

where

$$S_x = \hbar/2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5}$$

One bit of evidence that this is the right choice for S_x is provided by the eigenvalues of this matrix. I'll just write them down and you should check that they work:

$$|\pm\rangle_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \tag{6}$$

with eigenvalues $\pm\frac{\hbar}{2}$. This is as it should be. It shouldn't matter what we call the z, x or other axis; the possible eigenvalues should be the same. So the states $|\pm\rangle$ are the energy eigenstates, with eigenvalues: $E_\pm = \pm\frac{\hbar}{2}\frac{gB_x}{2m_e}$.

But now we can figure out what happens to our state. We expect that the spin should now precess about the x axis. So at different times, it should have different z component of the spin. To see how this works, we need to solve the Schrodinger equation. This is:

$$H\psi = i\hbar\frac{\partial\psi}{\partial t}. \tag{7}$$

The Schrodinger equation is linear. If we start out in the state $|+\rangle$, we have:

$$\psi(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-iE_+ t/\hbar} \tag{8}$$

Similarly if we start in the state $|-\rangle$. So if we start out in the state:

$$\psi(0) = \begin{pmatrix} a \\ b \end{pmatrix} \tag{9}$$

the solution of the Schrodinger equation at time t is:

$$\psi(t) = \frac{1}{\sqrt{2}}(a \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-iE_+t/\hbar} + b \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-iE_-t/\hbar}) \tag{10}$$

Returning to our original problem, we started out in the state

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\left(|+\rangle_x + |-\rangle_x\right) \tag{11}$$

To see what is going on, it helps to write this out. If we factor out $e^{-i\frac{(E_+ + E_-)t}{2\hbar}}$:

$$\psi(t) = \frac{1}{2}e^{-i\frac{(E_+ + E_-)t}{2\hbar}} \begin{pmatrix} e^{-i\frac{(E_+ - E_-)t}{2\hbar}} + e^{i\frac{(E_+ - E_-)t}{2\hbar}} \\ e^{-i\frac{(E_+ - E_-)t}{2\hbar}} - e^{i\frac{(E_+ - E_-)t}{2\hbar}} \end{pmatrix} \tag{12}$$

or

$$\psi(t) = \begin{pmatrix} \cos(\frac{(E_+ + E_-)t}{2\hbar}) \\ i\sin(\frac{(E_+ + E_-)t}{2\hbar}) \end{pmatrix} \tag{13}$$

So we see precisely the sort of precession we expect. If we measure the z component of spin at large times, we will sometimes find the spin pointing up, sometimes pointing down, even though we started with just up.

What does this have to do with neutrinos?

There are several types of neutrinos. For the problem of the sun, there are two which are important: the electron neutrino and the muon neutrino. Electron neutrinos are produced in β-decays of nuclei. They are produced in the nuclear reactions which power the sun. Muon neutrinos are found from decays of heavy particles called muons (we will talk about these more later). What is important, for now, is that the state of being an electron neutrino or a muon neutrino is like the state of being spin up or spin down. We can use the same notation:

$$|\nu_e\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} \qquad |\nu_\mu\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} \tag{14}$$

Now it turns out that having a neutrino mass is like having a magnetic field, partly in the z but partly in the x direction. This means that as time passes, an electron neutrino will sometimes have a chance of being found as a muon neutrino, and vice versa!