Physics 101B. Modern Physics. Professor Dine

Alpha Decay

1 Introduction

Half lives for particles which decay by α emission range from fractions of seconds to millions of years. They obey a striking relation:

$$\ln \tau_{1/2} = 144E^{-1/2} - 60.8 \tag{1}$$

where E is the energy of the outgoing α particle in the decay, measured in MeV, and τ is the half-life in years. We can derive this relation by a simple quantum mechanical analysis; along the way, we will encounter only one challenging integral.

2 A Model for α Decay

Because α particles are rather tightly bound, we can imagine the nucleus of charge Z and mass number A as consisting of $Z-2$ protons and $A-Z-4$ neutrons, bound to an α particle. The potential for the α particle we take to be:

$$V(r) = 0 \quad r < R; \qquad V(r) = \frac{2ke^2(Z-2)}{r} \quad r > R. \tag{2}$$

R is the size of the nucleus, $R \approx 1.2A^{1/3}$ Fm. The second term represents the electrostatic repulsion of the α particle and the nucleus. The typical energies in α decays a a few MeV. It is important that the height of the barrier is much larger; $V(R) \approx 50$ MeV.

Consider the Schrodinger equation for this system. To make things simple, assume $\ell = 0$ for the α particle. Were the barrier infinitely high, the α particle wave function would be confined to the well, and would behave as a sine. Because the barrier is not infinitely high, the particle "leaks out". The wave function does not vanish in the "forbidden region" where the potential is larger than the energy. The radial Schrodinger equation is:

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{2(Z-2)e^2K}{r}\right]\psi = E\psi. \tag{3}$$

Now we look for a solution of the form

$$\psi \approx e^{-A/\hbar}. \tag{4}$$

Plug back in the Schrodinger equation. Drop terms involving A''. These turn out to be a small correction which can be included in a more accurate analysis. Then we obtain:

$$\left[-\frac{1}{2m}A'^2 + \frac{2Ze^2k}{r} - E\right]\psi = 0 \tag{5}$$

so

$$A' = \sqrt{2m(V-E)}. \tag{6}$$

We can solve this for A. $V(r) - E > 0$ for $r > R$ and $r < r_1$ where

$$r_1 = \frac{2Ze^2K}{E}. \tag{7}$$

So if we want $A(r_1)$ $(\psi(r_1))$, we have:

$$A(r_1) = \sqrt{2m} \int_R^{r_1} dr \left(\frac{2Ze^2k}{r} - E \right)^{1/2} \tag{8}$$

This integral is not too hard to do. First rewrite this as:

$$A = \left(\frac{2mE}{\hbar^2} \right)^{1/2} \int_R^{r_1} \left(\frac{b}{r} - 1 \right)^{1/2} dr. \tag{9}$$

The integral can be evaluated by writing $r = r_a \cos^2 u$. The result is:

$$A = \left(\frac{2mE}{\hbar^2} \right)^{1/2} r_1 \left[\cos^{-1}(R/r_1)^{1/2} - \left(\frac{R}{r_1} \left(1 - \frac{R}{R_1} \right)^{1/2} \right) \right] \tag{10}$$

Now this can be brought to a nice form by noting $R/r_1 \ll 1$, so we can expand the cosine and the square root. The final result is:

$$A \approx \left(\frac{2mE}{\hbar^2} \right)^{1/2} r_1 \left[\frac{\pi}{2} - 2 \left(\frac{R}{r_1} \right)^{1/2} \right] \tag{11}$$

$$= 1.97ZE^{-1/2} - 1.49Z^{1/2}R^{1/2}.$$

Now we turn this into a lifetime by saying that the probability to find the α particle "over the barrier" is proportional to e^{-2A}, which is very small. The half-life is inversely proportional to this factor. It need to have dimensions of time, so we multiply by the time it takes like to cross the nucleus, about 3×10^{-22} s. This gives the formula for the lifetime above, as you should check!