Physics 101B. Modern Physics. Professor Dine

Momentum Wave Functions – Plane Waves

This is really a supplement to chapter six of your text. We especially want to understand section 6-4, particularly equations 6-48 and 6-49.

1 Plane Waves and Wave Packets

The problem of a particle in an infinite square well is easy to solve, but sines and cosines are a bit awkward to work with. It is easier to consider, instead, a particle subject to a different sort of boundary condition.

Start in one dimension. Then the periodic boundary condition is:

$$\psi(x+L) = \psi(x). \tag{1}$$

The Schrodinger equation is:

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\psi(x) \tag{2}$$

Calling $E = \frac{p^2}{2m}$, this has the solution:

$$\psi_p = e^{\frac{ipx}{\hbar}} \tag{3}$$

The periodicity requirement is the requirement that $pL = 2\pi n\hbar$.

Now in quantum mechanics, the operator $\hat{p} = -i\hbar\frac{d}{dx}$. So ψ_p satisfies:

$$\hat{p}\psi(x) = -i\hbar\frac{d}{dx}\psi = p\psi \tag{4}$$

so ψ is an eigenfunction of the momentum with eigenvalue p, as well as an eigenfunction of the energy with eigenvalue $\frac{p^2}{2m}$.

Before going to three dimensions, let's ask what is the typical value of n if $E = 1\ KeV$ and $L = 1$ meter. From our formulas,

$$E = \frac{2\pi^2\hbar^2 n^2}{2mL^2} \tag{5}$$

and solving for n, $n \approx 10^{10}$. So in practice, the spacing between states of different n is too small to measure. A real wave function for a particle in a box like this will be a sum (superposition) of states of different p. Since the n's are so large, we can replace the sum by an integral, and write:

$$\Psi(x) = \int dp\phi(p)e^{ipx/\hbar}e^{-i\frac{p^2}{2m}t/\hbar}. \tag{6}$$

As an exercise (you don't have to hand this in) you can check that $\psi(x)$ satisfies the time-dependent Schrodinger equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x) = i\frac{\partial}{\partial t}\Psi(x,t). \tag{7}$$

Now what's interesting here is that, just as $|\psi(x)|^2$ is the probability of finding the particle at a point x, so $|\phi|^2$ is the probability of finding the particle with momentum p.

The average value of many measurements of x is:

$$< x >= \int dx|\Psi|^2 x \tag{8}$$

and of many measurements of p is:

$$< p >= \int dp|\phi|^2 p \tag{9}$$

It is a theorem about Fourier transforms that the spread in the wave numbers ($k = p/\hbar$) is related to the spread in space by

$$\Delta k \Delta x \geq 1 \tag{10}$$

or $\Delta p \Delta x \geq \hbar$. This is the Heisenberg uncertainty principle. Note that for a pure plane wave, i.e. a state of definite momentum, the chance of finding the particle anywhere in space is a constant – there is no knowledge of the position.

A further fact about functions like this: as time passes, they move with speed $\frac{<p>}{m}$. They also spread out as time goes by.

In class, we generalized this to three dimensions. For a periodic box:

$$\psi_{\vec{p}} = e^{i\vec{p}\cdot\vec{x}}. \tag{11}$$

Periodicity requires:

$$\vec{p} = 2\pi(n_x, n_y, n_z) = 2\pi\vec{n} \tag{12}$$

$\psi_{\vec{p}}$ is an eigenfunction of the operator $\hat{\vec{p}} = -i\hbar\vec{\nabla}$ with eigenvalue $\vec{p}$ (since the momentum is a vector, there are really three operators with three different eigenvalues).

2 The Momentum Space Wave Function

Here we'll work in one dimension to keep things simple. The same ideas all carry over to three dimensions. Just as it doesn't make much sense in quantum mechanics to speak of a particle at a particular position, x, so we have just learned that it doesn't make much sense to speak of a particle of a definite momentum. Instead, we have just seen that a natural wave function for a particle moving in a big box is:

$$\psi(x,t) = \sum_n \frac{1}{\sqrt{L}} e^{ipx/\hbar} \phi(p) e^{-i\frac{p^2}{2m\hbar}t} \tag{13}$$

where $p = \frac{2\pi n\hbar}{\ell} = \hbar k$ (note that this is the de Broglie relation). Now $\phi(p)$, like ψ, has all the information we want about the particle. We can call this the *momentum wave function.* If $\phi(p)$ is a very narrow function of p, say peaked about p_o, we would guess that the most likely values of the momenta we would measure are around p_o. So we will guess, or postulate, that the probability of finding a particle with momentum $p + dp$ is

$$P(p)dp = |\phi(p)|^2 dp. \tag{14}$$

. This is just like $P(x) = |\psi(x)|^2$.

The average value of p is:

$$< p >= \sum_n |\phi(p)|^2 p \tag{15}$$

But now note that this is the same thing as:

$$\int dx \psi^*(x,t) - i\hbar \frac{\partial}{\partial x} \psi(x,t) \tag{16}$$

$$= \sum_n \sum_{n'} \int_o^L dx \frac{1}{L} \phi^*(p) e^{-ipx/\hbar} p e^{ip'x/\hbar} \phi^*(p') \tag{17}$$

$$= \sum_n p |\phi(p)|^2 \tag{18}$$

This is eqn. 6-48. The equation immediately below can be derived in the same way.

Suppose the wave function is a very peaked function of x, near some point x_o. Then $< x >= x_o$. A measure of how peaked the function is can be provided by the quantity

$$(\Delta x)^2 =< x^2 > - < x >^2 . \tag{19}$$

The more sharply peaked, the smaller this quantity will be. Similarly, one can define Δp. The Hesisenberg uncertainty principle, in this form, is a theorem about Fourier transforms. This theorem is important for any wave motion, e.g. for visible light. It says that the product of the spread in wave numbers (inverse wavelengths) and coordinates satisfies

$$\Delta x \Delta k > 1. \tag{20}$$

In terms of Δp, this is the Heisenberg uncertainty relation, which we introduced earlier.