Fall, 2005. Handout: The Kepler Problem

These notes are meant as a supplement to the materials in chapter 6 and 7 of your textbook.

1   The Kepler Problem

It is possible to solve the motion completely as a function of time. But for many purposes, we just want to know the shape of the orbits. In other words, we want to know $r(\theta)$ or $\theta(r)$. It is easiest to get $\theta(r)$. Start with

$$E = \frac{1}{2}\mu\dot{r}^2 + V_{eff} \tag{1}$$

where

$$V_{eff} = -\frac{k}{r} + \frac{\ell^2}{2\mu r^2}. \tag{2}$$

$$\frac{d\theta}{dr} = \frac{\dot{\theta}}{\dot{r}} = \frac{\ell dr}{r^2\sqrt{2\mu(E - V_{eff})}}. \tag{3}$$

So, for the Kepler problem,

$$\theta = \int^r \frac{dr\frac{\ell}{r^2}}{\sqrt{2\mu(E + k/r - \frac{\ell^2}{2\mu r^2})}}. \tag{4}$$

Change variables: $u = 1/r$. Then:

$$\theta(r) = -\int \frac{du}{(-u^2 + bu + c)^{1/2}} \tag{5}$$

with

$$b = \frac{2\mu k}{\ell^2} \qquad c = \frac{2\mu E}{\ell^2}$$

We can look up this integral in a table, but it is always good to be able to do things from scratch. Consider:

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \int \frac{dx}{\sqrt{a(x - x_+)(x - x_-)}} \tag{6}$$

$$= \int \frac{dx}{\sqrt{a(x - \alpha - \beta)(x - \alpha + \beta)}}$$

where $\alpha = \frac{-b}{2a}$; $\beta = \frac{\sqrt{b^2 - 4ac}}{2a}$, so we have do do:

$$\int \frac{dx}{\sqrt{a((x - \alpha)^2 - \beta^2)}} = \int \frac{du}{\sqrt{a(u^2 - \beta^2)}} \tag{7}$$

where $u = x - \alpha$. Finally, calling $u = \beta\cos(\theta)$, we obtain:

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = -\frac{1}{\sqrt{-a}}\sin^{-1}(\frac{2au + b}{\sqrt{b^2 - 4ac}}) \tag{8}$$

For our problem this gives:

$$\theta(r) = \sin^{-1}\left(\frac{-\frac{2}{r} + \frac{2\mu k}{\ell^2}}{\sqrt{\frac{4\mu^2 k^2}{\ell^4} + \frac{8\mu E}{\ell^2}}}\right) + C \tag{9}$$

Now we introduce some notation standard in astronomy:

$$\alpha = \frac{\ell^2}{\mu k} \qquad \epsilon = \sqrt{1 + \frac{2E\ell^2}{muk^2}} \tag{10}$$

α is called the "latus rectum"; ϵ the eccentricity. Take $c = -\frac{\pi}{2}$ (this defines the origin of θ); so

$$\sin(\theta + \pi/2) = \frac{1 - \frac{\alpha}{r}}{\epsilon}$$

or

$$\frac{\alpha}{r} = 1 + \epsilon \cos(\theta) \tag{11}$$

Every high school student knows this is an ellipse! Of course, I am not a high school student, so I have to do some work. First, a special case: $\epsilon = 0$ is a circle ($r = \alpha$). Note that in this case $E = -\frac{\mu k^2}{2\ell^2}$, as expected.

Second, let's draw the curve, for $\epsilon = .2484, .967$.

Finally, let's cast this in the form we all learned in high school. Start with:

$$\alpha = r + \epsilon x$$

or

$$r = \alpha - \epsilon x.$$

So, squaring,

$$x^2 + y^2 = \alpha^2 + \epsilon^2 x^2 - 2\alpha\epsilon x$$

Now we rearrange this a bit, completing the squares:

$$x^2(1 - \epsilon^2) - 2\alpha\epsilon x + y^2 = \alpha^2$$

We can rearrange this as:

$$\frac{(x - x_o)^2}{a^2} + \frac{(y - y_o)^2}{b^2} = \alpha^2 \frac{1}{1 - \epsilon^2} \tag{12}$$

For a general potential, an interesting question is: Are the orbits closed? We can examine $\Delta\theta$, the change in the polar angle in one passage between the turning points, r_{min} and r_{max}. In order that the orbit be closed, it is necessary that $\Delta\theta$ be a rational multiple of π. From our previous formulae,

$$\Delta\theta = 2 \int_{r_{min}}^{r_{max}} \frac{dr \frac{\ell}{r^2}}{\sqrt{2\mu(E + \frac{k}{r} - \frac{\ell^2}{2\mu r^2})}}.$$

This is only a multiple of π for the case of a $1/r$ and an r^2 potential.

Let's check this for the Kepler problem. It is easiest to work with the variable u. The turning points correspond to the points where $\dot{r} = 0$ in the expression:

$$E = \frac{1}{2}\mu\dot{r}^2 - \frac{k}{r} + \frac{\ell^2}{2\mu r^2} \tag{13}$$

Multiplying by $\frac{2\mu E}{\ell^2}$, this gives, at the turning points:

$$c = -\frac{b}{r} + \frac{1}{r^2} \tag{14}$$

or

$$au^2 + bu + c = 0. \tag{15}$$

This gives

$$\theta(r_+) = \frac{\pi}{2} + C \qquad \theta(r_-) = -\frac{\pi}{2} + C.$$

So in one complete revolution, θ changes by 2π.

2   Solution of the Problem by Perturbation Theory

The procedure above is rather opaque. We have the exact solution, but limited insight into what is going on. We can solve the problem, however, by the methods of perturbation theory. Like the pendulum, this is a non-linear problem. There we did a perturbation theory in the amplitude of the oscillation. Here, we know how to solve the problem of circular orbits. We can solve for orbits which are nearly circular, again, by perturbation theory.

Recall that for a circular orbit,

$$r_o = \frac{L^2}{\mu k}. \tag{16}$$

r_o, of course, is constant. For an orbit which is nearly circular, we write:

$$r(t) = r_o + \delta r(t). \tag{17}$$

The equation of motion for r is:

$$\ddot{r}(t) = -\frac{\partial V_{eff}}{\partial r} = -\frac{k}{r(t)^2} + \frac{L^2}{2\mu r(t)^3}. \tag{18}$$

If $\delta r(t) \ll r_o$, we can Taylor expand the function on the right hand side of this equation in powers of δr. Let's keep the first two terms:

$$\mu \ddot{\delta r} + \frac{d^2 V}{dr^2}|_{r=r_o} \delta r = -\frac{1}{2} \frac{d^3 V}{dr^3}|_{r=r_o} \delta r^2. \tag{19}$$

If we just keep the first term, we have a harmonic oscillator equation for δr,

$$\ddot{\delta r} + \omega_o^2 \delta r = 0. \tag{20}$$

Here,

$$\omega_o = \left(\frac{1}{\mu} \frac{d^2 V}{dr_o^2} \right)^{1/2} = \frac{k^2 \mu}{L^3}. \tag{21}$$

We found earlier that the motion is strictly periodic, i.e. both r and θ should have the same period. We see here that they do;

$$\frac{2\pi}{\dot{\theta}} = \frac{2\pi \mu r_o^2}{L} = \frac{2\pi L^3}{\mu k^2}, \tag{22}$$

vs.

$$\frac{2\pi}{\omega_o} = \frac{2\pi L^3}{\mu k^2}. \tag{23}$$

We can also see that the motion is an ellipse. Note that $r_o = \alpha$, the parameter we introduced earlier to describe the ellipse. Also note, from $\dot{\theta}$, that

$$\theta = \frac{\mu k^2}{L^3 t},\tag{24}$$

so the curve is

$$r = r_o + A\cos(\frac{k^2\mu}{L^3}t + \delta) = \alpha - A\cos(\theta),\tag{25}$$

or

$$\frac{\alpha}{r} = 1 + \frac{A}{r_o}\cos(\theta).\tag{26}$$

This, we saw, is the equation for an ellipse with $\epsilon = A/r_o$ (we are assuming that A, and hence the eccentricity, is very small).

We can keep going, including the δr^2 terms in the equation. When we studied the pendulum, we saw that the non-linearity had two effects: there is a correction to the frequency, and higher harmonics appear. The dependence on the frequency depended on on the amplitude, θ_o. The non-linear correction here also leads to a change in the frequency (and hence the period). We can evaluate this as before, and compare with the exact expression for the period, equation 8.46. I'll leave the algebra for you. You will want to reexpress the energy in 8.46 in terms of the eccentricity, 8.40. There are also higher harmonics (you may want to think of this graphically, or in terms of the motion in the r potential).