Fall, 2005. Handout: Systems of Particles

Fall, 2005. Also Problem Set 8. Due Mon., Nov. 28

These notes are meant as a supplement to the materials in chapter 9 of your textbook. We are going to go very quickly through this chapter. This is intended to give you an overview.

1 Multiple Particles: Conservation Laws, Lagrangian

1.1 Conservation of Momentum: Newton's Equations

We can derive the conservation laws two ways. First, from Newton's laws. Suppose we have N particles, with masses m_a and coordinates $\vec{x}_a$. We consider the case of central forces. In terms of potentials, this means $V = V(|\vec{x}_a - \vec{x}_b|)$. We also allow for the possibility of an external force (e.g. electric field). In terms of forces, this means that the force on particle a is:

$$\vec{f}_a = \sum_{b \neq a}^{N} \vec{f}_{ab} + \vec{F}_a \tag{1}$$

The last term represents the external force on the particle. For central forces, Newton's third law holds, $\vec{f}_{ab} = -\vec{f}_{ba}$.

We generalize our definition of the center of mass for two particles:

$$\vec{R} = \frac{1}{M} \sum_{a=1}^{N} m_a \vec{x}_a. \tag{2}$$

where $M = \sum m_a$ is the total mass. For a continuous distribution, this becomes:

$$\vec{R} = \frac{1}{M} \int d^3 x \rho(\vec{x}) \vec{x}. \tag{3}$$

Let's find the equation satisfied by $\vec{R}$:

$$\ddot{\vec{R}} = \frac{1}{M} \sum_{a=1}^{N} m_a \frac{d^2 \vec{x}_a}{dt^2} = \frac{1}{M} \sum_{a=1}^{N} \vec{f}_a \tag{4}$$

$$= \frac{1}{M} \left(\sum_{a=1}^{N} \sum_{b \neq a} \vec{f}_{ab} + \sum_{a=1}^{N} \vec{F}_a \right).$$

The first term vanishes by Newton's third law. The last term is the net external force. So the cm coordinate obeys:

$$\frac{d\vec{P}}{dt} = \vec{F}. \tag{5}$$

If there is no external force, the system just moves with constant velocity.

1.2 Conservation of Momentum: Lagrangian Description

We have already proven conservation of momentum in the lagrangian description. We have also seen how to go to CM and relative coordinates for two particles. For more than two particles, in either the Newtonian or Lagrangian description, the transition to center of mass coordinates is a bit more involved. One way to proceed is to proceed iteratively. Take particles 1 and 2. Define their center of mass and relative coordinates, $\vec{R}_{12}$ and r_{12}. Now consider particle three, and take the center of mass and relative coordinates of the system $\vec{R}_{12}, \vec{x}_3$. And so on to particle four.

Explicitly for three particles:

$$\vec{r}_{12} = \vec{x}_1 - \vec{x}_2; \vec{R}_{12} = \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{m_{12}}; \mu_{12} = \frac{m_1 m_2}{m_{12}} \tag{6}$$

where $m_{12} = m_1 + m_2$. Then one defines:

$$\vec{\rho} = \vec{R}_{12} - \vec{x}_3; \vec{R} = \frac{m_{12}\vec{R}_{12} + m_3\vec{x}_3}{m_3 + m_{12}} \tag{7}$$

Problem 1. Show that in terms of these variables, with no external forces, the lagrangian is:

$$L = \frac{1}{2}M\left(\frac{d\vec{R}}{dt}\right)^2 + \frac{1}{2}\mu\left(\frac{d\vec{\rho}}{dt}\right)^2 + \frac{1}{2}\mu_{12}(\frac{d\vec{r}_{12}}{dt})^2 - V(|\vec{r}_{12}|, |\vec{\rho}|) \tag{8}$$

where $\mu = \frac{m_{12}m_3}{m_{12}+m_3} = \frac{m_{12}m_3}{M}$. Show that the total momentum of the system is conserved, and construct the Hamiltonian.

1.3 Angular Momentum

In terms of the original coordinates, the angular momentum is:

$$\vec{L} = \sum_a \vec{x}_a \times \vec{p}_a = \sum_a m_a(\vec{x}_a \times \frac{d\vec{x}_a}{dt}). \tag{9}$$

Now call $\vec{x}_a = \vec{R} + \vec{r}_a$. So

$$\vec{L} = \sum m_a(\vec{R} + \vec{x}_a) \times (\frac{d\vec{R}}{dt} + \frac{d\vec{r}_a}{dt}) \tag{10}$$

$$= \sum m_a \vec{R} \times \frac{d\vec{R}}{dt} + \sum m_a \vec{R} \times \frac{d\vec{r}_a}{dt} + \sum m_a \frac{d\vec{R}}{dt} \times \vec{r}_a + \sum m_a \vec{r}_a \times \frac{d\vec{r}_a}{dt}$$

The two terms in the middle vanish, since $\sum m_a \vec{r}_a = 0$. So the angular momentum is the sum of the center of mass momentum and the relative momentum. This form is obvious from our lagrangian description.

1.4 Returning to the Lagrangian Description

We have seen that passing to center of mass coordinates for more than two particles is a somewhat awkward process. We can proceed in a somewhat different way, using the lagrangian and using lagrange multipliers.

Consider the coordinates, $\vec{r}_a$ we introduced above. We saw that they obey the constraint:

$$\sum m_a \vec{r}_a = 0. \tag{11}$$

So let's rewrite the lagrangian in terms of $\vec{R}$ and $\vec{r}_a$, being mindful of the constraint.

$$L = \sum_{a=1}^{N} \frac{1}{2} m_a \left(\frac{d(\vec{r}_a + \vec{R})}{dt} \right)^2 + \vec{\lambda} \cdot \left(\sum_{a=1}^{N} m_a \vec{r}_a \right) - V(\{\vec{r}_a\}) \tag{12}$$

As before, the cross terms in the time derivative vanish due to the constraint (Note, by the way, that there are really three constraints, so $\vec{\lambda}$ is a vector). We can solve for $\vec{\lambda}$ by examining the equations of motion for the $\vec{r}_a$'s:

$$m_a \frac{d^2 \vec{r}_a}{dt^2} = -\vec{\nabla}_a V + m_a \vec{\lambda} \tag{13}$$

From the constraint, we see that if we multiply each equation by m_a and sum, we get:

$$M \vec{\lambda} = -\sum_a \vec{f}_a \tag{14}$$

but we have already seen that the right hand side is zero if there are no external forces acting on the system. So $\vec{\lambda} = 0$. So we actually now have a quite simple lagrangian:

$$L = \sum_{a=1}^{N} \frac{1}{2} m_a \left(\frac{d\vec{r}_a}{dt} \right)^2 + \frac{1}{2} M \left(\frac{d\vec{R}}{dt} \right)^2 - V(|\vec{r}_a|) \tag{15}$$

where we have to keep in mind that there are N, not $N + 1$ independent degrees of freedom, due to the constraint.

Problem 2. Determine $\vec{\lambda}$ in the case that there is an external force.

2 Scattering in the Center of Mass

We have seen how to separate the center of mass and the relative motion for two particles. Before considering many particles, let's consider some further aspects of the center of mass motion for two particles. We can consider a reference frame which moves with the center of mass:

$$\vec{x}_{cm} = \vec{x} - \vec{V}_{cm} t \tag{16}$$

Suppose we have, in the lab, two particles colliding. They might interact through electromagnetic forces or gravitational forces (we could be thinking about the collision of a comet with a planet). In the lab ("lab frame") suppose that the particles, with masses m and M, have momenta $\vec{p}$ and $\vec{0}$. Then the center of mass velocity is:

$$\vec{V} = \frac{d}{dt} \vec{R} = \frac{m}{M + m} \frac{\vec{p}}{m} = \frac{\vec{p}}{m + M}. \tag{17}$$

Note that the momenta in the center of mass frame are:

$$\vec{p}_1 = \vec{p} - \frac{m}{m + M} \vec{p} = \frac{M}{m + M} \vec{p} \qquad \vec{p}_2 = -\frac{M}{m + M} \vec{p} \tag{18}$$

i.e. the momenta are equal and opposite; the total momentum in the center of mass frame is zero.

For elastic scattering, then, the kinematics of collisions are very simple. We start with two equal and opposite momenta. Momentum conservation then says that after the collision we have equal and opposite momenta. So the only thing we have to describe in scattering is an *angle*.

Let's consider a special case: "back scattering", ie. a head on collision. Let's also consider the case $M \gg m$. Then

$$\vec{V} \approx \frac{m\vec{v}}{M} = \frac{\vec{p}}{M}. \tag{19}$$

In the center of mass frame, the heavy object has momentum $\vec{p}$, and moves to the left. After the collision, it moves with the same momentum to the right. In the lab frame, its momentum is now $2\vec{p}$. The incoming particle moves to the left with velocity $-\frac{\vec{p}}{m} + \frac{\vec{p}}{M} \approx -\frac{\vec{p}}{m}$. So momentum is conserved; the light particle slows down slightly, and the heavy particles moves slowly to the right.

Alternatively, consider the heavy particle incident on the light particle. Now the velocity of the center of mass is nearly that of the heavy particle, $\vec{v}$. In the center of mass, we now have:

$$\vec{p}_1 = -m\vec{v} \tag{20}$$

(the heavy particle is nearly at rest). After the collision, the light particle has momentum, in the center of mass,

$$\vec{p} = 2m\vec{v} \tag{21}$$

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Problem 3. Keep all of the terms in the velocity, i.e. don't make the approximation of very small mass. Work out the momenta in both frames and check that momentum is conserved in each frame.

3   Rutherford Scattering

Scattering experiments are a big part of physics. Perhaps the prototype, and one of the most important, was the experiment carried out by Rutherford and his assistants scattering α particles (obtained from radioactive decays) on gold foils.

Consider scattering from a very massive particle. (For a light particle, there is an equivalent problem in the center of mass frame). The impact parameter determines the scattering angle θ. Also, the angular momentum, with respect to the origin defined by the scattering center, is:

$$\ell = mvb = b\sqrt{2mT_o}. \tag{22}$$

If we were just scattering off, say, a hard sphere or a disk, the chance of a particle scattering is proportional to the cross section of the object. If j is the current density – the number of particles passing per unit area per second – the number scattered per second is σj. Generally we want to know, for a scattering, the number of particles scattered per second per target particle into a small element of solid angle, $d\Omega$. We define the differential cross section by:

$$d\Omega \frac{d\sigma(\theta)}{d\Omega} = \frac{\text{(No. of scatterings/target particle into } d\Omega)}{\text{No. of incident particles/unit area}} \tag{23}$$

If we have azimuthal symmetry (as we do for Rutherford scattering),

$$d\Omega = 2\pi d\theta \sin(\theta) \tag{24}$$

For an impact parameter b, particles are scattered to angle $\theta(b)$. For particles entering in a range of b, $b, b + db$, we have scattering into $\theta + d\theta$.

$$2\pi b db = -\frac{d\sigma}{d\Omega} 2\pi \sin\theta d\theta \tag{25}$$

So

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin(\theta)} \left| \frac{db}{d\theta} \right| \tag{26}$$

Now for a central potential, we can get the relation between θ and b from our expression for the angle as a function of r. Actually, what we called θ before is not quite the angle which enters in the scattering problem. Call the angle we used earlier Θ; then

$$\theta = \pi - 2\Delta\Theta \tag{27}$$

(the factor of two comes from our definition of Θ as the angle between r_{min} and r_{max}).

Our previous formula was:

$$\Delta\Theta = \int_{r_{min}}^{r_{max}} \frac{(1/r^2)dr}{\sqrt{2m(E - V - \frac{\ell^2}{2\mu r^2})}} \tag{28}$$

$r_{max} = \infty$. To work out r_{min}, we need to figure out the distance of closest approach. From

$$V = \frac{k}{r} \qquad k = \frac{q_1 q_2}{4\pi\epsilon_o} \tag{29}$$

we have, for r_{min}:

$$E - \frac{k}{r_{min}} - \frac{\ell^2}{2\mu r_{min}^2} = 0. \tag{30}$$

Rewriting the integral:

$$\Delta\Theta = \int_{r_{min}}^{\infty} \frac{(b/r^2)dr}{\sqrt{1 - b^2/r^2 - V/T_o}} \tag{31}$$

We can do the integral as before.

Problem 4. Fill in the details of the integration; get the cross section formula below to the end.

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We obtain:

$$\cos(\Theta) = \frac{\kappa}{b} \frac{1}{\sqrt{1 + (\kappa/b)^2}} \qquad \kappa = \frac{k}{2\,T_o}. \tag{32}$$

Squaring:

$$\cos^2(\Theta) = \frac{\frac{\kappa^2}{b^2}}{1 + \frac{\kappa^2}{b^2}} \tag{33}$$

or

$$\frac{\kappa^2}{b^2} = \frac{1}{\tan^2(\Theta)} \tag{34}$$

Finally, using the relation between Θ and θ,

$$b = \kappa \cot(\theta/2). \tag{35}$$

So

$$\frac{db}{d\theta} = -\frac{\kappa}{2}\frac{1}{\sin^2(\theta/2)}. \tag{36}$$

and

$$\frac{d\sigma}{d\theta} = \frac{-\kappa}{2}\frac{1}{\sin^2(\theta/2)}. \tag{37}$$

Putting all of this together:

$$\frac{d\sigma}{d\Omega} = \frac{k^2}{4T_o}\frac{1}{\sin^4(\theta/2)} \tag{38}$$

A few features of the cross section are interesting. First, the total cross section is obtained by integrating over angles. But this is infinite:

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = 2\pi \int d\theta \frac{d\sigma}{d\Omega}. \tag{39}$$

This reflects the fact that the Coulomb force has infinite range. In the real world, this is shielded by other charges.

Rutherford thought that the atom was a smooth charge distribution. So he initially thought that inside the atom, the potential would go to zero. He expected to see almost no back-scattered particles. Instead, he saw many. From this he realized that the atom in fact has a small nucleus, and measured its size.

4 Centrifugal and Coriolis Forces

Newton's laws refer to an idealized "inertial frame." But in many real situations, we are not in such an inertial frame, and Newton's laws don't hold. An example is the surface of the earth. For many purposes, the earth's rotation ($\omega = 7.3 \times 10^{-5}$rad/sec) is unimportant. But there are situations where it is (the weather?), and other situations where rotational effects are important. These effects can be described in the non-inertial frame in terms of two fictitious forces: the centrifugal force and the coriolis force.

To begin, we consider the motion of an object fixed in the rotating frame, as viewed from the inertial frame. Remembering our discussion of infinitesimal rotations,

$$(d\vec{r})_{fixed} = d\vec{\theta} \times \vec{r} \tag{40}$$

so dividing by dt,

$$\left(\frac{d\vec{r}}{dt}\right)_{fixed} = \vec{\omega} \times \vec{r} \tag{41}$$

where $\vec{\omega} = \frac{d\vec{\theta}}{dt}$ is the angular velocity of the rotation. Now if the point moves with respect to the origin of the rotating system:

$$\left(\frac{d\vec{r}}{dt}\right)_{fixed} = \left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\omega} \times \vec{r}. \tag{42}$$

For a general vector:

$$\left(\frac{d\vec{Q}}{dt}\right)_{fixed} = \left(\frac{d\vec{Q}}{dt}\right)_{rot} + \vec{\omega} \times \vec{Q}. \tag{43}$$

If the origin also has some translational motion relative to the fixed frame, writing $\vec{r}' = \vec{R} + \vec{r}$,

$$\left(\frac{d\vec{r}'}{dt}\right)_{fixed} = \left(\frac{d\vec{R}}{dt}\right)\left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\omega} \times \vec{r}. \tag{44}$$

Now we want to see what Newton's second law looks like from the point of view of the rotating frame. Call

$$\vec{v}_f = \left(\frac{d\vec{r}'}{dt}\right)_{fixed} \qquad \vec{V} = \left(\frac{d\vec{R}}{dt}\right)_{fixed}$$

$$\vec{v}_r = \left(\frac{d\vec{r}}{dt}\right)_{rot}$$

then

$$\vec{v}_f = \vec{V} + \vec{v}_r + \vec{\omega} \times \vec{r}$$

Starting with $\vec{F} = m\vec{a}_{fixed}$, differentiating $\vec{v}_f$ with respect to time, and using our relations between the various fixed and rotating quantities gives, after a little algebra::

$$\vec{F} = m\ddot{\vec{R}}_f + m\vec{a}_r + m\dot{\vec{\omega}} \times \vec{r} + m\vec{\omega} \times (\vec{\omega} \times \vec{r}) + 2m\vec{\omega} \times \vec{v}_r \tag{45}$$

So the rotating observer thinks there is a force:

$$\vec{F}_{eff} = m\vec{a}_r = \vec{F} - m\ddot{\vec{R}}_f - m\dot{\vec{\omega}} \times \vec{r} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2m\vec{\omega} \times \vec{v}_r \tag{46}$$

In many situations, the $\dot{\omega}$ term is negligible. The last two terms are the centrifugal force and the Coriolis force, respectively.

4.1 Foucault pendulum

Here we consider the motion of a pendulum. The vertical is the z direction. The horizontal are the x and y directions. The equations of motion are:

$$\ddot{x} + \omega_o^2 x = 2\omega_z \dot{y} \quad \ddot{y} + \omega_o^2 y = -2\omega_z \dot{x} \tag{47}$$

where ω_z is the component of the earth's rotation in the z direction, and $\omega_o^2 = \frac{g}{\ell}$. To solve, we note that typically $\omega_z \ll \omega_o$. So to first approximation, we can neglect ω_z. So we have decoupled oscillators. Over large times, we expect that the plane of oscillation will rotate slowly. So we look for a solution of the form:

$$x = d\cos(\phi(t))\sin(\omega_o t + \delta) \quad y = d\sin(\phi(t))\sin(\omega_o t + \delta). \tag{48}$$

Here ϕ changes slowly with time. Now we can evaluate the first and second derivatives of x and y, keeping only first derivative terms in ϕ. Plugging in the x, y equations of motion gives:

$$-2d\omega_o\frac{d\phi}{dt}\sin(\phi(t))\cos(\omega_o t + \delta) \approx 2\omega_z\omega_o\sin(\phi(t))\cos(\omega_o t + \delta) \tag{49}$$

So

$$\frac{d\phi}{dt} = -\omega_z. \tag{50}$$

So

$$\phi = -\omega_z t \tag{51}$$

(taking $\phi = 0$ at $t = 0$). So

$$x = d\cos(\omega_z t)\sin(\omega_o t + \delta) \quad y = -d\sin(\omega_z t)\sin(\omega_o t + \delta) \tag{52}$$

Problem 5. Verify the equations above for the pendulum, starting with the equation of motion.

Additional Problems

Problem 6. 9-1.

Problem 7. 9-36.

Problem 8. 9-46.

Problem 9. 10-8.

Problem 10. 10-12.