Physics 105. Mechanics. Professor Dine

Fall, 2004. Handout: The Central Field Problem

These notes are meant as a supplement to the materials in chapter 8 of your textbook.

1 Lagrangian and Hamiltonian in Special Relativity

A guess:

$$S = -mc\int ds = -mc\int\sqrt{c^2dt^2 - d\vec{x}^2} \tag{1}$$

$$= \int dtL$$

$$L = -mc^2\sqrt{1-(\frac{1}{c^2}\frac{d\vec{x}}{dt})^2} \tag{2}$$

Note that for small velocities,

$$L = -mc^2 + \frac{1}{2}m\dot{x}_i^2 \tag{3}$$

so up to a constant, it is the lagrangian we have used for non-relativistic problems. Using the Hamiltonian construction, the momenta are:

$$p^i = \frac{\partial L}{\partial \dot{x}^i} = \frac{m\dot{x}^i}{\sqrt{1-v^2/c^2}} \tag{4}$$

The Hamiltonian is:

$$H = p^i\dot{x}^i - L \tag{5}$$

$$= \frac{mv^2 + mc^2(1-\frac{1}{c^2}v^2)}{\sqrt{1-v^2/c^2}}$$

$$= \frac{mc^2}{\sqrt{1-v^2/c^2}}$$

2 Pendulum with Lagrange Multipliers

$$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + mgy - \lambda(x^2+y^2-\ell^2). \tag{6}$$

We can solve this by writing: $x = \ell\sin(\theta); y = -\ell\cos(\theta)$. But let's proceed with the lagrange multipliers. The equations of motion for x and y are:

$$m\ddot{x} + 2\lambda x = 0; m\ddot{y} - mg + 2\lambda y = 0. \tag{7}$$

The variation with respect to λ gives the constraint:

$$x^2+y^2-\ell^2 = 0. \tag{8}$$

Differentiating twice gives:

$$\ddot{x}x + \ddot{y}y + \dot{x}^2 + \dot{y}^2 = 0. \tag{9}$$

We can substitute this in the equations for x and y. In particular, if we take the first equation, and multiply by y, and the second and multiply by x and add, we get:

$$m(x\ddot{x} + y\ddot{y}) + 2\lambda(x^2 + y^2) + mgy = 0. \tag{10}$$

Using the constraint equation, we can eliminate the first term, and solve for λ:

$$\lambda = \frac{-mgy + m(\dot{x}^2 + \dot{y}^2)}{2(x^2 + y^2)}. \tag{11}$$

This is, in fact, the tension in the rod. If you subsitute back, e.g., in the x equation, you see the force T_x.

$$m\ddot{x} - \frac{mgxy}{x^2 + y^2} + \frac{m(\dot{x}^2 + \dot{y}^2)x}{(x^2 + y^2)} \tag{12}$$

i.e. $T_x = -T\sin(\theta)$ and similarly for y.