These notes are meant as a supplement to the materials in chapter 11 of your textbook. We are going to go very quickly through this chapter. This is highlight a few key points.

We saw that in general, the moment of inertia is given by:

\[ I_{ij} = \int d^3x \rho(\vec{x})(\delta_{ij} \vec{x}^2 - x_i x_j). \]  

(1)

In terms of this tensor, the angular momentum is given by:

\[ (L_{rot})_i = I_{ij} \omega_j \]  

(2)

and the kinetic energy by:

\[ T_{rot} = \frac{1}{2} I_{ij} \omega_i \omega_j \]  

(3)

For some simple cases:

1. Disk:

\[ I_{zz} = \int_0^R drr \int_0^{2\pi} \frac{M}{\pi R^2} r^2 \, dr = \frac{M}{2} R^2. \]  

(4)

similarly:

\[ I_{xx} = I_{yy} = \frac{M}{4} R^2 \]  

(5)

As a matrix:

\[ I = \frac{MR^2}{2} \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]  

(6)

2. Ring:

\[ I_{zz} = MR^2 \]  

(7)

Sphere:

\[ I_{xx} = I_{yy} = I_{zz} = \frac{2}{5} MR^2. \]  

(8)

As a matrix:

\[ I = \frac{2}{5} MR^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]  

(9)

We made judicious choices of axes in each case so that \( I \) is diagonal. For a general choice of axes, this will not be the case. To see this, note that \( I_{ij} \) is a tensor. Under rotations,

\[ x_i = O_{ij} x'_j \]  

(10)
where $O$ is a rotation matrix ($O^TO = 1$). So

$$I_{ij} = O_{ik}O_{jl}I'_{kl}$$

(11)

which is the same transformation law as that of $x_ix_j$; this is the definition of a tensor.

Now we can think what happens to our various tensors under rotations. For the sphere,

$$I_{ij} = O_{ik}I_{kl}O^T_{lj}$$

(12)

is invariant, because $I$ is proportional to the unit matrix, and $O$ is orthogonal. This is not surprising; there is no preferred axis. This is also the case for the disk or the ring, provided we do rotations only about the axis of symmetry.

If we are given a complicated $I$, with off diagonal entries, we can always do a rotation to a set of axes for which the matrix $I$ is diagonal, and the rotations relatively simple. These are called the “principal axes.” The idea (for this I will have to refer you to Boas, section 10.3,10.4 ) is to find the matrix, $O$, which diagonalizes $I$. The steps are:

1. Solve the characteristic equation, $\det(I - \lambda) = 0$.
2. This yields three eigenvalues, $\lambda_i$. For each eigenvalue, solve the equation:

$$I_{ij}\vec{\omega} = \lambda\vec{\omega}$$

The $\vec{\omega}$’s are the principle axes (the rotations are simple about these). As explained in Boas, the $O$ matrices are simple; they are essentially:

$$O = (\vec{\omega}_1 \vec{\omega}_2 \vec{\omega}_3).$$

(13)

The moment of inertia is worked out in your text. One of the principle axes, for example, is along the diagonal. To see this, start with:

$$I = Ma^2 \begin{pmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{pmatrix}$$

(14)

We need to solve:

$$\det Ma^2 \begin{pmatrix} 2/3 - \lambda & -1/4 & -1/4 \\ -1/4 & 2/3 - \lambda & -1/4 \\ -1/4 & -1/4 & 2/3 - \lambda \end{pmatrix} = 0$$

(15)

We can simplify this by defining

$$-\frac{\lambda'}{4} = 2/3 - \lambda.$$

So the determinant is proportional to:

$$\det \begin{pmatrix} \lambda' & 1 & 1 \\ 1 & \lambda' & 1 \\ 1 & 1 & \lambda' \end{pmatrix} = \lambda'^3 - 3\lambda' + 2$$

(16)

Fortunately, this factorizes to $(\lambda' + 2)(\lambda' - 1)^2$. So the roots are $\lambda' = -2$, $\lambda' = 1$ (twice). So the $\lambda$’s are 11/12 and 1/6.

The eigenvector with $\lambda' = -2$ satisfies:

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = 0$$

(17)

which is satisfied if the $\omega_i$’s are all equal. This is one of the diagonals. You can work out the other cases similarly.