Physics 105. Mechanics. Professor Dine

Fall, 2005. Handout: Moment of Inertia

These notes are meant as a supplement to the materials in chapter 11 of your textbook. We are going to go very quickly through this chapter. This is highlight a few key points.

We saw that in general, the moment of inertia is given by:

$$I_{ij} = \int d^3x \rho(\vec{x})(\delta_{ij}\vec{x}^2 - x_i x_j). \tag{1}$$

In terms of this tensor, the angular momentum is given by:

$$(L_{rot})_i = I_{ij}\omega_j \tag{2}$$

and the kinetic energy by:

$$T_{rot} = \frac{1}{2} I_{ij}\omega_i\omega_j \tag{3}$$

For some simple cases:

1. Disk:

$$I_{zz} = \int_o^R drr \int_o^{2\pi} d\theta (\frac{M}{\pi R^2}) r^2 = \frac{M}{2} R^2. \tag{4}$$

similarly:

$$I_{xx} = I_{yy} = \frac{M}{4} R^2 \tag{5}$$

As a matrix:

$$I = \frac{MR^2}{2} \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6}$$

2. Ring:

$$I_{zz} = MR^2 \tag{7}$$

Sphere:

$$I_{xx} = I_{yy} = I_{zz} = \frac{2}{5} MR^2. \tag{8}$$

As a matrix:

$$I = \frac{2}{5} MR^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{9}$$

We made judicious choices of axes in each case so that I is diagonal. For a general choice of axes, this will not be the case. To see this, note that I_{ij} is a tensor. Under rotations,

$$x_i = O_{ij} x'_j \tag{10}$$

where O is a rotation matrix ($O^T O = 1$). So

$$I_{ij} = O_{ik} O_{jl} I'_{kl} \tag{11}$$

which is the same transformation law as that of $x_i x_j$; this is the definition of a tensor.

Now we can think what happens to our various tensors under rotations. For the sphere,

$$I_{ij} = O_{ik} I_{kl} O^T_{lj} \tag{12}$$

is invariant, because I is proportional to the unit matrix, and O is orthogonal. This is not surprising; there is no preferred axis. This is also the case for the disk or the ring, provided we do rotations only about the axis of symmetry.

If we are given a complicated I, with off diagonal entries, we can always do a rotation to a set of axes for which the matrix I is diagonal, and the rotations relatively simple. These are called the "principal axes." The idea (for this I will have to refer you to Boas, section 10.3,10.4) is to find the matrix, O, which diagonalizes I. The steps are:

1. Solve the characteristic equation, $\det(I - \lambda) = 0$.

2. This yields three eigenvalues, λ_i. For each eigenvalue, solve the equation:

   $$I_{ij} \vec{\omega} = \lambda \vec{\omega}$$

   The $\vec{\omega}$'s are the principle axes (the rotations are simple about these). As explained in Boas, the O matrices are simple; they are essentially:

   $$O = (\vec{\omega}_1 \ \vec{\omega}_2 \ \vec{\omega}_3). \tag{13}$$

The moment of inertia is worked out in your text. One of the principle axes, for example, is along the diagonal. To see this, start with:

$$I = Ma^2 \begin{pmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{pmatrix} \tag{14}$$

We need to solve:

$$\det Ma^2 \begin{pmatrix} 2/3 - \lambda & -1/4 & -1/4 \\ -1/4 & 2/3 - \lambda & -1/4 \\ -1/4 & -1/4 & 2/3 - \lambda \end{pmatrix} = 0 \tag{15}$$

We can simplify this by *defining*

$$-\frac{\lambda'}{4} = 2/3 - \lambda.$$

So the determinant is proportional to:

$$\det \begin{pmatrix} \lambda' & 1 & 1 \\ 1 & \lambda' & 1 \\ 1 & 1 & \lambda' \end{pmatrix} = \lambda'^{\,3} - 3\lambda' + 2 \tag{16}$$

Fortunately, this factorizes to $(\lambda' + 2)(\lambda' - 1)^2$. So the roots are $\lambda' = -2, \lambda' = 1$ (twice). So the λ's are $11/12$ and $1/6$.

The eigenvector with $\lambda' = -2$ satisfies:

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = 0 \tag{17}$$

which is satisfied if the ω_i's are all equal. This is one of the diagonals. You can work out the other cases similarly.