Physics 105. Mechanics. Professor Dine

Fall, 2005. Handout: Oscillations

These notes are meant as a supplement to the materials in chapter 3 of your textbook.

1 Axions: The Dark Matter of the Universe? A Damped Harmonic Oscillator?

Over the past few years, a great deal of evidence has accumulated that most of the energy density of the universe is in some form other than protons and neutrons (ordinary matter). It is not known what this might be. One suggestion is that it is a new kind of particle, known as an "axion." To understand how the axion behaves in the early universe, let's first solve the problem of the damped harmonic oscillator again, with a small damping. Rather than solve it exactly, we will solve it approximately, using the idea that the oscillator oscillates, but that the amplitude of oscillation slowly decreases with time.

Starting with

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = 0 \tag{1}$$

and assuming $\beta \ll \omega_o$, we look for a solution of the form:

$$x(t) = f(t)\cos(\omega_o t). \tag{2}$$

Then

$$\dot{x} = -\omega_o f \sin(\omega_o t) + \dot{f}\cos(\omega_o t) \qquad \ddot{x} = -\omega_o^2 f \cos(\omega_o t) - 2\omega_o \dot{f}(t)\sin(\omega)_o t) + \ddot{f}\cos(\omega_o t). \tag{3}$$

Now we plug this back into the original equation, but keep only the fewest derivatives of f. This gives:

$$-2\omega_o \dot{f}(t)\sin(\omega_o t) - 2\omega_o \beta f(t)\sin(\omega_o t) = 0 \tag{4}$$

or

$$\dot{f} = -\beta f \qquad f = a e^{-\beta t}. \tag{5}$$

So this is the solution we found by proceeding exactly.

Now the axion obeys a similar equation, but now with a time-dependent friction (which comes, it turns out, from the expansion of the universe):

$$\ddot{x} + \frac{2}{t}\dot{x} + \omega_o^2 x = 0. \tag{6}$$

Proceeding as before:

$$\dot{f} = -\frac{1}{t} f \tag{7}$$

so

$$f = \frac{a}{t}. \tag{8}$$

2 Fourier series and the Driven Oscillator

(See, for example, chapter 7 of Boas).

For functions which are periodic with period T, calling $\omega = \frac{2\pi}{T}$,

$$f(t) = \sum_{-\infty}^{\infty} a_n e^{in\omega t} \tag{9}$$

This form is in some ways the simplest; it can also be written in terms of sines and cosines. The a_n's can then be written quite compactly:

$$a_n = \frac{1}{T}\int_o^T dt f(t) e^{-in\omega t}. \tag{10}$$

Using this we can write a solution for any periodic forcing function:

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = f(t) = \sum a_n e^{in\omega t} \tag{11}$$

by using the superposition principle for linear differential equations. For any one exponential, we have:

$$x = x_{comp} + \sum_{-\infty}^{\infty} \frac{a_n}{(\omega_o^2 - \omega^2 n^2) + 2in\beta\omega} \tag{12}$$

This approach, however, is limited to periodic forcing functions. Using Fourier *transformations* we can consider more general driving terms (see Boas, p. 648). Any function which vanishes sufficiently rapidly at infinity can be written as:

$$f(t) = \int_{-\infty}^{\infty} g(\omega) e^{i\omega t} d\omega \tag{13}$$

with

$$g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \tag{14}$$

Again, using the linear superposition principle, we can write the general solution of our differential equation, now in terms of an integral rather than a sum:

$$\ddot{x} + 2\beta\dot{x} + \omega_o^2 x = f(t) = \int g(\omega) e^{i\omega t} d\omega \tag{15}$$

so

$$x = x_{comp} + \int_{-\infty}^{\infty} \frac{g(\omega)}{(\omega_o^2 - \omega^2) + 2i\beta\omega} e^{i\omega t} \tag{16}$$

An example where we can do everything in closed form is a gaussian,

$$f(t) = e^{-\frac{t^2}{t_o^2}} \tag{17}$$

Then

$$g(\omega) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{-i\omega t - \frac{t^2}{t_o^2}} \tag{18}$$

Now the good thing about Gaussian integrals like this is that you can always do them. We can derive a formula which always works (even when some of the constants are complex, as here):

$$\int dx e^{-ax^2 + bx} = \int dx e^{-a(x - \frac{b}{2a})^2 + \frac{b^2}{4a}} \tag{19}$$

Now change variables, $u = x - b/2$, and the integral becomes:

$$e^{\frac{b^2}{4a}} \int_{-\infty}^{\infty} e^{-au^2} \tag{20}$$

$$= \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}$$

For us, $a = \frac{1}{t_o^2}$, $b = -i\omega$, so

$$g(\omega) = t_o \sqrt{\pi} e^{-\frac{\omega^2 t_o^2}{4}}. \tag{21}$$

So now we can write the solution to the differential equation as an integral. For the particular solution we have:

$$x_{part} = \int_{-\infty}^{\infty} d\omega \frac{t_o \sqrt{\pi} e^{-\frac{\omega^2 t_o^2}{4}}}{(\omega_o^2 - \omega^2) + 2i\beta\omega} e^{i\omega t} \tag{22}$$

Now this integral, I have to confess, is hard. But it simplifies in the limit that $\beta \ll \omega$. Then the integral gets most of its contribution from the region where $\omega = \omega_o$. In fact, if β were zero, the integral would be infinite!

$$x_{part} \approx \int_{-\infty}^{\infty} d\omega \frac{t_o \sqrt{\pi} e^{-\frac{\omega_o^2 t_o^2}{4}}}{(\omega_o^2 - \omega^2) + 2i\beta\omega_o}. \tag{23}$$

The remaining integral can be done in various ways (e.g. by contour integrals – later in 114B)

$$= \frac{i}{\omega_o} t_o \sqrt{\pi} e^{-\frac{\omega_o^2 t_o^2}{4}} e^{i\omega_o t - \beta t}$$