Physics 105. Mechanics. Professor Dine

Fall, 2005. Handout: The Pendulum

These notes are meant as a supplement to the materials in chapter 4 of your textbook.

1 The Pendulum: Solution By Perturbative Methods

The equation of motion for the pendulum is

$$\ddot{\theta} + \omega_o^2 \sin(\theta) = 0. \tag{1}$$

This is a highly non-linear equation. For small oscillation, however, it is nearly linear. The usual elementary treatment corresponds to approximating

$$\sin(\theta) \approx \theta. \tag{2}$$

Then the pendulum oscillates with frequency ω_o, independent of the amplitude. We can, however, consider corrections, even if the oscillations are small. These introduce some interesting features. Here we keep the next correction to the sine function:

$$\sin(\theta) \approx \theta - \frac{1}{6}\theta^3. \tag{3}$$

The equation of motion is now:

$$\ddot{\theta} + \omega_o^2 \theta = \frac{1}{6}\omega_o^2 \theta^3. \tag{4}$$

We could try to solve this equation by approximating θ on the right hand side by the lowest order solution,

$$\theta = \theta_o \cos(\omega_o t) \tag{5}$$

(we have taken as initial conditions $\theta(0) = \theta_o; \dot{\theta}(0) = 0$). But we have to be careful. Calling $\theta(t) = \theta_o \cos(\omega_o t) + \delta(t)$, the equation for $\delta(t)$ is:

$$\ddot{\delta}(t) + \omega_o^2 \delta(t) = \frac{\omega_o^2}{6}\theta_o^3 (\frac{3}{4}\cos(\omega_o t) + \frac{1}{8}\cos^3(\omega_o t)). \tag{6}$$

We are making the approximation of dropping terms of higher order in θ_o than θ_o^3. This procedure – known as perturbation theory – has produced a linear equation for $\delta\theta$. This is an equation for a harmonic oscillator with two driving terms, and we can try to find a particular solution by solving for $\delta\theta$ with each term separately. The problem is that the first term is a driving term at exactly the *resonant frequency*. So we can't solve this in the usual way. Instead, we make an educated guess. We assume that the frequency is corrected, i.e. that

$$\theta(t) = \theta_o \cos(\omega_1 t) + \delta(t) \tag{7}$$

where ω_1 is not much different than ω_o. Then the equation with the first forcing term gives an equation for ω_1:

$$(\omega_o^2 - \omega_1^2)\theta_o = \frac{\omega_o^2 \theta_o^3}{8}\omega_o^2, \tag{8}$$

or, since $\omega_o \approx \omega_1$,

$$\omega_1 - \omega_o = \frac{1}{16}\omega_o \theta_o^2. \tag{9}$$

We can derive this result another way, using conservation of energy. We will do this in a moment, but we first put this together to get the full result. Looking for a solution

$$\delta(t) = A\cos(3\omega_o t) \tag{10}$$

we find, plugging back in:

$$A = \frac{\theta_o^3}{388} \tag{11}$$

The figure shows a comparison with a numerical solution of the equation and the analytic result, for different values of θ_o.

Finally, let's work out another derivation of the frequency. We use conservation of energy. For the pendulum, the energy is:

$$E = \frac{1}{2}\ell m\dot{\theta}^2 + mg\ell(1 - \cos(\theta)). \tag{12}$$

The energy is conserved. For a pendulum starting at rest at θ_o,

$$E = mg\ell(1 - \cos(\theta_o). \tag{13}$$

So we can write:

$$\frac{d\theta}{dt} = \sqrt{2mg\ell(1 - \cos(\theta_o) - 1 + \cos(\theta))} = \sqrt{2mg\ell(\cos(\theta) - \cos(\theta_o))}. \tag{14}$$

In the usual way, we can rewrite this as:

$$\int_o^{\frac{T}{4}} dt = \int_o^{\theta_o} d\theta \frac{1}{\sqrt{2mg\ell(\cos(\theta) - \cos(\theta_o))}} \tag{15}$$

I won't do the integral here, but you can check that if you expand the cosine and keep the leading term, you obtain the result we found above.