Physics 110. Electricity and Magnetism. Professor Dine

Spring, 2008. Handout: Radiation

1 Electric Dipole Radiation

Our treatment in class was essentially that of section 11.1.4, radiation from an arbitrary source. We derived 11.51 and 11.52, as well as the $\vec{E}$ and $\vec{B}$ fields and the Poynting vector and power. Let's summarize briefly the derivation of the fields V and $\vec{A}$. $\vec{A}$ is particularly easy:

$$\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int d^3x' \frac{1}{|\vec{x}-\vec{x}'|}\vec{J}(\vec{x}',t_r). \tag{1}$$

For point particles, labelled by a,

$$\vec{J} = \sum \vec{v}_a q_a \delta(\vec{x}-\vec{x}_a(t)) \tag{2}$$

If the particles are moving slowly with respect to the speed of light, we can do the integral without being very careful about t_r. For the a'th particle,

$$t_r = t - |\vec{x}-\vec{x}_a(t)|/c. \tag{3}$$

Also, for an observation point far away from the collection of particles,

$$|\vec{x}-\vec{x}_a| \approx r \tag{4}$$

so

$$\vec{A}(\vec{x},t) \approx \frac{\mu_0}{4\pi}\sum q_a \frac{\vec{v}_a(t-r/c)}{r}. \tag{5}$$

So we see we have an outgoing spherical wave. The result may be written in terms of the time derivative of the dipole moment of the system,

$$\vec{A}(\vec{x},t) \approx \frac{\mu_0}{4\pi}\frac{\frac{d\vec{p}(t-r/c)}{dt}}{r} \tag{6}$$

We can evaluate V similarly, but we can use, instead, the Lorentz gauge condition. We have

$$\vec{\nabla}\cdot\vec{A} = -\frac{\mu_0}{4\pi c}\frac{\ddot{\vec{p}}(t-r/c)\cdot\hat{r}}{r} \tag{7}$$

So the equation

$$\mu_0\epsilon_0\frac{dV}{dt} + \vec{\nabla}\cdot\vec{A} = 0 \tag{8}$$

is solved by

$$V = \frac{1}{4\pi\epsilon_0 c}\frac{\frac{d\vec{p}(t-r/c)}{dt}\cdot\hat{r}}{r}. \tag{9}$$

Now we can compute the $\vec{E}$ and $\vec{B}$ fields very easily, and quite generally. The point is that when we take derivatives with respect to r, *we get terms of order* $1/r^2$, except from differentiating the $t - r/c$ terms in the argument of $\vec{p}$. So using

$$\partial_i r = \hat{r}_i \tag{10}$$

we obtain

$$\vec{B} = -\frac{\mu_0}{4\pi c}\hat{r} \times \frac{d^2\vec{p}}{dt^2} \tag{11}$$

and

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left(\frac{1}{c^2}\frac{d^2\vec{p}}{dt^2}\cdot\hat{r}\;\hat{r} - \epsilon_0\mu_0\frac{d^2\vec{p}}{dt^2}\right) \tag{12}$$

and

$$S = \frac{1}{c^2(4\pi\epsilon_0)^2 r^2}\left[\hat{r}\left(\left(\frac{d^2\vec{p}}{dt^2}\right)^2 - (\frac{d^2\vec{p}}{dt^2}\cdot\hat{r})^2\right)\right] \tag{13}$$

2 Magnetic Dipole Radiation

We can derive a general expression for magnetic dipole radiation in a similar way. This radiation will be important for systems for which the electric dipole moment vanishes.

The calculation is simplified by some tricks, so it is useful to do the static case (i.e. constant current) first. The

$$\vec{A} = \frac{\mu_o}{4\pi}\int \frac{d^3x'\,\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}. \tag{14}$$

The leading term in powers of $1/r$ should vanish, since we don't have any magnetic monopoles in our problem. To see this, take the large r limit:

$$A_i = \frac{\mu_o}{4\pi r}\int d^3x'\,J_i(\vec{x}') \tag{15}$$

$$= \frac{\mu_o}{4\pi r}\int d^3x'\nabla'_k(x'_i J_k).$$

The last equation follows by just taking the derivatives, and using $\vec{\nabla}\cdot\vec{J} = 0$. But the integral of a divergence is zero, so the leading term vanishes.

So we need to be more careful, and expand up one term.

$$\vec{A}_i = \frac{\mu_o}{4\pi r^2}\int d^3x'(\hat{x}_j x'_j)J_i(\vec{x}'). \tag{16}$$

We do a similar trick. Note that

$$\int d^3x'\nabla'_k(x'_i x'_l J_k) = 0 \tag{17}$$

$$= \int d^3x'(x'_i J_l + x'_l J_i)$$

So

$$A_i = -\frac{\mu_o\hat{x}_k}{4\pi r^2}\int d^3x'\frac{1}{2}[x'_i J_k - x'_k J_i]. \tag{18}$$

Defining

$$\vec{m} = \frac{1}{2}\int d^3x'\vec{x}'\times\vec{J}(\vec{x}') \tag{19}$$

it follows that

$$\vec{A} = \frac{\mu_o}{4\pi}\frac{\vec{m}\times\vec{x}}{|\vec{x}|^3}. \tag{20}$$

Special case:

$$\vec{J} = \sum q_a\vec{v}_a\delta(\vec{x}-\vec{x}^2). \tag{21}$$

Then

$$\vec{m} = \sum \frac{q_a \vec{L}_a}{2m_a}. \tag{22}$$

Similarly, get standard result for a wire.

Now, radiation:

$$\vec{A} \approx \frac{\mu_o}{4\pi r} \int d^3x' \vec{J}(\vec{x}', t_R). \tag{23}$$

As in the case of the wire, we need to be careful about t_r:

$$t_r \approx t - \frac{r}{c} + \frac{\vec{x}' \cdot \hat{r}}{c} \tag{24}$$

If we drop the last term, we get zero from the integral over d^3x', just as we did in the constant current case. So we need to keep one more term. This gives:

$$A_i = \frac{\mu_o}{4\pi rc} \hat{r}_i \int d^3x' \hat{r}_k \hat{x}'_k \dot{J}_i. \tag{25}$$

This is just the sort of expression we encountered before, and gives:

$$\vec{A} = \frac{\mu_o}{4\pi rc} \hat{r} \times \frac{d\vec{m}}{dt}. \tag{26}$$

3 Thomson Scattering

Consider a wave incident on an electron. The electron could be free or bound to an atom. If the electric field of the wave is:

$$\vec{E} = \hat{x} E_o e^{ikz - i\omega t} \tag{27}$$

then the electron experiences a force due to the field; calling the electron charge e, and supposing that it is bound harmonically, the equation of motion is:

$$m\ddot{x} + Kx = eE_o e^{-i\omega t}. \tag{28}$$

(I have taken the electron to sit at the origin and I have only written the equation for the x coordinate). We know the solution (dropping the homogeneous solution, e.g. assuming small damping):

$$x = \frac{eE_o}{m(\omega^2 - \omega_o^2)}. \tag{29}$$

From this, we can immediately write the dipole moment:

$$\vec{p} = \hat{x} \frac{e^2 E_o}{m(\omega_o^2 - \omega^2)}. \tag{30}$$

We can plug this into our formula for the radiated energy. But what we would actually like to do is divide by the incoming flux. This gives the fraction of energy scattered into any given angle. We also want to consider unpolarized light, which means we want to average over polarization initially in the $\hat{x}$ and $\hat{y}$ direction. The incoming flux is

$$\vec{S}_{inc} = \hat{z} \frac{1}{2} \epsilon_o E_o^2 \tag{31}$$

while the outgoing flux is (because of the averaging, and taking $\omega_o = 0$):

$$\vec{S}_{scattered} = \hat{r} \frac{\mu_o}{16\pi^2 m^2 r^2 c} e^4 E_o^2 \frac{1}{2} \left[(\hat{r} \times \hat{x})^2 + (\hat{r} \times \hat{y})^2 \right]; \tag{32}$$

the term in the brackets is: $\frac{1}{2}(1+\cos^2\theta)$. Taking the ratio give:

$$\frac{d\sigma}{d\Omega} = \frac{\mu_o e^4}{16\pi^2 m^2 c\epsilon_o}(1+\cos^2\theta). \tag{33}$$

This is a famous formula, the Thomson formula for scattering of light by charged particles. Had we kept ω_o, we would have obtained Rayleigh scattering.

We can think of this from the point of view of quantum mechanics. There we are interested in an incoming photon beam, and the fraction of photons scattered at each angle. But this is what we calculated above, since the energy is proportional to the photon frequency, and this is the same for the incoming and scattered waves. In quantum mechanics, this scattering process is called Compton scattering, and the result agrees with the classical result only for small frequencies ($\hbar\omega \ll m_e c^2$).

The quantum mechanical formula (known as the Klein-Nishima formula) is:

$$\frac{d\sigma}{d\cos(\theta)} = c\frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right]. \tag{34}$$

In the limit of small ω (incoming photon frequency), $\omega' \approx \omega$, and this becomes the Thomson formula.

If you want to match the constants, you need:

$$\alpha = \frac{e^2}{4\pi\hbar c\epsilon_o}. \tag{35}$$