Physics 110. Electricity and Magnetism. Professor Dine

Spring, 2008. Handout: Special Relativity

1 Lorentz Transformations

Consider, first, boosts along the x axis. These should preserve:

$$c^2t^2 - x^2. \tag{1}$$

If we call $x^o = ct$, this is similar to the conservation of vector lengths by ordinary rotations, except for the funny minus sign. So we write the transformation in the form:

$$x = \cosh(\omega)x' + \sinh(\omega)x^{o\prime} \tag{2}$$

$$x^o = \sinh(\omega)x' + \cosh(\omega)x^{o\prime}.$$

Exercise: Check that this conserves $(x^o)^2 - x^2$.

Solution:

$$(x^o)^2 - x^2 = (\sinh(\omega)x' + \cosh(\omega)x^{o\prime})^2 - (\cosh(\omega)x' + \sinh(\omega)x^{o\prime})^2$$

Just working out the terms, this is equal to

$$(x^{o\prime})^2(\cosh^2 - \sinh^2) - x'^2(\cosh^2 - \sinh^2)$$

$$= (x^{o\prime})^2 - x'^2$$

which is what we want to show (we have used the identity for the hypergeometric functions $\cosh^2 - \sinh^2 = 1$).

In order that we reproduce our time-dilation formula, we must have

$$\cosh(\omega) = \gamma \tag{3}$$

from which it follows that $\sin(\omega) = v\gamma$ (so that $\cosh^2 - \sinh^2 = 1$).

This analogy with rotations suggests generalizing the notion of a vector (something which transforms in a definite way under rotations) to a four-vector (an object which transforms in a definite way under Lorentz transformations). Calling

$$x^\mu = (ct, \vec{x}) = (x^o, \vec{x}) \tag{4}$$

we have the rule:

$$x^o = \gamma(v/cx' + x^{o\prime}). \tag{5}$$

$$x = \gamma(x' + v/cx^{o\prime}).$$

It is not hard to generalize these rules to boosts by a velocity $\vec{v}$ in an arbitrary direction:

$$x^o = \gamma(\frac{1}{c}\vec{v} \cdot x' + x^{o\prime}). \tag{6}$$

$$\vec{x} = \hat{v}\gamma(\hat{v} \cdot x' + v/cx^{o\prime}) + (\vec{x}' - \hat{v}\vec{x}' \cdot \hat{v}).$$

Here $\hat{v} = \vec{v}/v$.

Exercise: Check that this gives the correct form for transformations along the x axis. Check that for any transformation, the components of a vector perpendicular to the velocity are unchanged.

Solution: With $\vec{v} = v\hat{x}$, these formulas become:

$$\vec{x}^o = \gamma(x_1' + x^{o\prime})$$

$$\vec{x}_1 = \hat{x}_1' + vx^{o\prime}$$

$$\vec{x}_2 = \vec{x}_2'$$

$$\vec{x}_3 = \vec{x}_3'$$

The vectors x^μ, and any vectors which transform in the same way, are called *contravariant* vectors. It is convenient to define covariant vectors,

$$x_\mu = (-x^o, \vec{x}). \tag{7}$$

Note that $x^\mu x_\mu = \vec{x}^2 - c^2t^2$. This is the quantity preserved by Lorentz transformations. We can write the transformations in a matrix form:

$$x^\mu = \Lambda^\mu_\nu x^{\nu\prime}. \tag{8}$$

Exercise: Work out the components of Λ

Solution: These appear in your book and class notes.

Just like $O^TO = 1$ for rotations, there is a similar relation for the matrices Λ.

2 The Proper Time

In special relativity, $x_\mu y^\mu$ is invariant. A particularly important four vector is the differential, dx^μ. This transforms like any other four vector; it is, after all, the difference of two four vectors.

From the differential, we can construct an invariant

$$dx^\mu dx_\mu = -c^2 d\tau^2$$

$$= -c^2dt^2 + d\vec{x}^2$$

In a particle's rest frame, $d\tau$ is just the time which elapses. So $d\tau$ is called the "proper time." While it refers to a special frame, it is a Lorentz-invariant notion.

A useful way to write the proper time in a general frame is:

$$c^2 d\tau^2 = -c^2dt^2(1 - \frac{1}{c^2}\left(\frac{d\vec{x}}{dt}\right)^2) \tag{9}$$

or $d\tau = \gamma dt$. We see again the time dilation effect.

3 Lagrangian and Hamiltonian in Special Relativity

We can try to write a lagrangian for a free particle. In order that the equations of motion for the particle take the same form in any frame, we can try to find a lagrangian which is *Lorentz Invariant*. We have seen that $d\tau$ is Lorentz invariant. So we try, as action, the integral over $d\tau$, multiplied by a constant, which we call $-mc^2$.

$$S = -mc^2 \int d\tau = -mc^2 \int dt \sqrt{1 - v^2/c^2} \tag{10}$$

$$= \int dtL$$

$$L = -mc^2 \sqrt{1 - (\frac{1}{c^2}\frac{d\vec{x}}{dt})^2} \tag{11}$$

Note that for small velocities,

$$L = -mc^2 + \frac{1}{2}m\dot{x}_i^2 \tag{12}$$

so up to a constant, it is the lagrangian we have used for non-relativistic problems. Using the Hamiltonian construction, the momenta are:

$$p^i = \frac{\partial L}{\partial \dot{x}^i} = \frac{m\dot{x}^i}{\sqrt{1 - v^2/c^2}} \tag{13}$$

The Hamiltonian is:

$$H = p^i\dot{x}^i - L \tag{14}$$

$$= \frac{mv^2 + mc^2(1 - \frac{1}{c^2}v^2)}{\sqrt{1 - v^2/c^2}}$$

$$= \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

So we have derived $E = \gamma mc^2$ just following our rules from Physics 105.

We are really supposed to express the Hamiltonian in terms of the momenta. This is easy to do here, and gives the important relation:

$$H^2 = E^2 = m^2c^4 + p^2c^2. \tag{15}$$

4 Four-Velocity and Four-Momentum

We can construct another four-vector by dividing dx^μ by $d\tau$. The result is the four-velocity:

$$u^\mu = \frac{dx^\mu}{d\tau} \tag{16}$$

with components:

$$u^o = \gamma c; u^i = \gamma v^i. \tag{17}$$

The components of u^μ are not all independent;

$$u^\mu u_\mu = -(u^o)^2 + \vec{u}^2 = \gamma^2(v^2 - c^2) = -c^2. \tag{18}$$

This is a consequence of the definition of u^μ.

The four velocity is closely related to another four-vector, the *energy-momentum four-vector*,

$$p^\mu = mu^\mu. \tag{19}$$

The components are:

$$p^o = mc\gamma; p^i = mv^i\gamma. \tag{20}$$

At low velocities, p^i is the ordinary three-momentum. p^o is E/c, the relativistic energy divided by c.

The constraint on u^μ is closely related to the energy-momentum relation:

$$(p^o)^2 - \vec{p}^2 = m^2c^2. \tag{21}$$

We see that this is a *relativistically invariant* relation.

One very valuable fact to note: γ is the energy divided by the mass of the particle. This is almost always the easiest way to find the γ factor for the motion of a particle. If you absolutely need to know the velocity for some reason, you can then solve for v in terms of γ. (E.g. SLAC; high energy gamma rays).

5 More Four-Vectors

Another very useful four-vector is the gradient. We write this as:

$$\partial_\mu = \frac{\partial}{\partial x^\mu}. \tag{22}$$

This transforms under Lorentz transformations as a *covariant vector* (indices downstairs). One way to see this is to note that

$$\partial_\mu x^\mu = 4 \tag{23}$$

i.e. it is a number, which is Lorentz invariant. But otherwise, you can use the chain rule to check.

With this observation,

$$\Box = \partial_\mu \partial^\mu = \vec{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial^2} \tag{24}$$

is *Lorentz invariant*!

Note, also, that the energy-momentum in quantum mechanics is a four vector:

$$p^\mu = i\hbar\partial^\mu. \tag{25}$$

(Check that this form, with the i instead of $-i$, is correct).

Still another useful Lorentz invariant is the *current*, j^μ. It's components are:

$$j^o = c\rho; \vec{j} = \vec{J}. \tag{26}$$

I won't prove it now, but let's think about it. If we have a charge at rest, then under a boost, we have a charge density and a current. The charge density is increased by a γ factor as a result of the Lorentz contraction. The current is the charge density times the velocity. As one further piece of evidence, this makes the equation of current conservation,

$$\partial_\mu j^\mu = 0 \tag{27}$$

a Lorentz-invariant equation, i.e. true in any frame.

6 The Vector Potential as a Four-Vector

If we define a four-vector A^μ by

$$A^o = cV; \vec{A} = \vec{A} \tag{28}$$

then the Lorentz gauge condition is:

$$\partial_\mu A^\mu = c\frac{\partial V}{\partial t} + \vec{\nabla} \cdot \vec{A} = 0. \tag{29}$$

In other words, it is a Lorentz-invariant condition. Moreover, in this gauge, the equations for the potentials can be written in a *manifestly Lorentz-covariant* form:

$$\Box A^\mu = j^\mu. \tag{30}$$

7 Mechanics again

So far, we wrote the lagrangian for a free particle. We can try and guess a lagrangian for a particle in an electromagnetic field from. We might expect that the lagrangian should involve V. It should also be Lorentz covariant. So we try:

$$S = -mc \int d\tau + \frac{1}{c}\int dx^\mu A_\mu. \tag{31}$$

At low velocities, the last term is $-\int dtV$. In general, we can write this as:

$$S = \int dt(-mc\gamma - qV + \frac{q}{c}\vec{A} \cdot \vec{v}). \tag{32}$$

Our goal is to show that this lagrangian gives the Lorentz force law. We need to work carefully with the Euler-Lagrange equations. First, we work out:

$$\frac{\partial L}{\partial \dot{x}_i} = \gamma m v_i + \frac{q}{c}A_i. \tag{33}$$

To write the equations of motion, we need to take the *total derivative* with respect to time. A_i has, in general, both explicit time dependence and time dependence coming from its dependence on $x_j(t)$:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{d\tau}{dt}m\frac{d}{d\tau}u + \frac{q}{c}\dot{A}_i + \frac{q}{c}v_j\partial_j A_i. \tag{34}$$

In the Euler-Lagrange equations, we also need:

$$\frac{\partial L}{\partial x_i} = -q\partial_i V + \frac{q}{c}(\partial_i A_j)v_j. \tag{35}$$

Putting this all together:

$$\frac{1}{\gamma}m\dot{u} = -q(\frac{1}{c}\dot{A}_i + \partial_i V) - \frac{q}{c}v_j(\partial_j A_i - \partial_i A_j) \tag{36}$$

$$= q\vec{E} + \frac{q}{c}\vec{v} \times \vec{B}.$$

This is the relativistic generalization of the Lorentz force law:

$$m\ddot{u} = \gamma q[\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}]. \tag{37}$$

8 More on the Matrices Λ

Now that we are a bit more used to four vector notation, let's return to Λ. Write:

$$x^\mu = \Lambda^\mu_\nu x^{\nu\prime}; \; y_\mu = y'_\rho \tilde{\Lambda}^\rho_\mu. \tag{38}$$

So in order that $y_\mu x^\mu$ be invariant, we need:

$$\tilde{\Lambda}^\rho_\mu \Lambda^\mu_\nu = \delta^\rho_\nu. \tag{39}$$

This is the case for our simple Lorentz transformation along the z axis:

$$\Lambda = \begin{pmatrix} \gamma & \gamma\frac{v}{c} & 0 & 0 \\ \gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{40}$$

$$\tilde{\Lambda} = \begin{pmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{41}$$

(note here that to make the matrices easy to write, I have taken the four vector x^μ to be:

$$\begin{pmatrix} x^o \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}. \tag{42}$$

It is true for the more general transformation we have written above.

In terms of Λ, it is easy to understand the covariance of equations. For example,

$$\partial^2 A^\mu = j^\mu \tag{43}$$

is covariant, since both sides transform in the same fashion under Lorentz transformations.

9 Electrodynamics in a Relativistic Notation

You may have seen the transformation laws for the electric and magnetic fields in the past. They are rather complicated in appearance. This is because the electric and magnetic fields transform into each other. There are six fields in total, so they can't be a four vector. They are, in fact, a tensor:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{44}$$

In particular,

$$F_{oi} = -E_i; \; F_{ij} = -B_i. \tag{45}$$

These transform as:

$$F^{\mu\nu} = \Lambda^\mu_\rho \Lambda^\nu_\sigma F^{\rho\sigma}. \tag{46}$$

There is not much to do but multiply these out. You find the transformation laws for the $\vec{E}$ and $\vec{B}$ fields in your text.

What about the Lorentz force law? This should involve $\dot{u}^\mu$, which is a four-vector which would generalize the acceleration. It should be linear in the fields $\vec{E}$ and $\vec{B}$, i.e. linear in $F^{\mu\nu}$. A guess is:

$$m\dot{u}^\mu = q u_\nu F^{\mu\nu}. \tag{47}$$

To see this is right, we need to check the components.

A fitting place to end: everything (almost) looks like electrodynamics.

Strong interactions: $F^a_{\mu\nu}$, $a = 1, \dots 8$. Equations for the fields:

$$\partial_\nu F^{\mu\nu a} = J^{\mu a}. \tag{48}$$

Currents: *color* of the quarks.
Weak interactions: same thing! ($a = 1 \dots 3$). Currents: weak charge. But includes an extra piece, due to the *Higgs field*, which leads to a mass:

$$m^2 A^{i2}_\mu.$$