Physics 110. Electricity and Magnetism. Professor Dine

Spring, 2008. Handout: A Wave Review

Generalities about Waves

The utility of the cosine/sine solutions of the wave equation can be understood by thinking about the Fourier expansion of a wave. Physicists speak of "plane waves", since the surfaces on which the waves are constant (in space) are planes (e.g. $z = C$ for a wave moving along the z axis). Such a wave carries infinite energy; real waves are bounded in space and carry finite energy. Still, provided that the waves are not too narrow in space, the plane wave picture is a good one. More precisely, if Δx is the spread in space, the spread in wave number, Δk, is of order

$$\Delta k \sim \frac{1}{\Delta x}$$

So if $\Delta x \gg \lambda$, the typical wavelength, the spread in k is small compared to a wavelength.

Let's focus on motion in one dimension. Fourier's integral theorem (see Boas, chapter 15, for example) states that a general function, defined on the line, can be expanded, under suitable conditions, as:

$$f(x) = \int_{-\infty}^{\infty} g(k) e^{ikx} dk \tag{1}$$

where the function $g(k)$ is given by

$$g(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \tag{2}$$

This can be understood as a limit of the ordinary Fourier series, written, not in terms of sines and cosines, but of exponentials:

$$f(x) = \frac{2\pi}{L} \sum_n e^{\frac{2\pi i n}{L}} g_n(2\pi/L) \tag{3}$$

where

$$g_n = \frac{1}{2\pi} \int_{-L/2}^{L/2} e^{\frac{-2\pi i n}{L}} f(x) \tag{4}$$

(here I have changed slightly the *definition* of g_n, multiplying it by a constant compared to what Boas uses; the content of the formulas is the same). Taking the limit $L \to \infty$, and calling $k = 2\pi n/L$, one can replace the sum over n by an integral, and note

$$dn = L \frac{dk}{2\pi} \tag{5}$$

Exercise: Verify that eqns. 3 and 4 agree with what you learned in 116 (e.g. check Boas, or your favorite text; make sure you understand why that factors of 2π and L are questions of convention, i.e. check how this convention compares with Boas). Then verify that (1) and (2) follow as a limiting case.

The problems that usually interest us in physics satisfy the "suitable conditions": electromagnetic signals carrying finite energy, Schrodinger wave functions which have finite probability ("normalizable").

Let's consider a particularly simple example: a wave which has a Gaussian shape. In other words, remembering that the solution of the wave equation which moves to the right is

$$f(x,t) = f(x - vt) \tag{6}$$

We'll take $g(k)$ to be a nice Gaussian peaked about a particular wave number, k_0:

$$g(k) = \frac{1}{\Delta k} \exp\left(-\frac{(k-k_0)^2}{(\Delta k)^2}\right) \tag{7}$$

We have learned that solutions of the wave equation are functions $f(x - vt)$. So to determine $f(x,t)$ we need to evaluate:

$$f(x - vt) = \int_{-\infty}^{\infty} dk \frac{1}{\Delta k} \exp\left(-\frac{(k-k_0)^2}{(\Delta k)^2} + i(kx - \omega t)\right) \tag{8}$$

To do this integral, we just need the following "integral table":

$$\int_{-\infty}^{\infty} du e^{-au^2+bu} \tag{9}$$

$$= \int_{-\infty}^{\infty} du e^{-a(u-b/2a)^2+\frac{b^2}{4a}}$$

$$= \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}.$$

For us

$$a = 1/(\Delta k)^2; \quad b = \frac{2k_0}{(\Delta k)^2} + i(x - vt).$$

So plugging in the formula:

$$f(x,t) = \sqrt{\pi} e^{ik_0 x - \omega_0 t} e^{-(x-vt)^2 \frac{\Delta k^2}{4}}. \tag{10}$$

This is almost the plane wave; it is essentially the plane wave but localized in a region of space centered at $x = vt$, of size of order $1/\Delta k$.

Boundary Conditions

Let's consider the case of two strings tied together at $x = 0$. The tensions and mass per unit length are different on each side. Call the solution to the left f_-, the solution to the right f_+. If the wave comes in from the left, then f_- will be the sum of an incoming wave (moving to the right) and a reflected wave (moving to the left). f_+ will describe a transmitted wave: We can write this as:

$$f_- = A_I e^{ik_1(x-v_1t)} + A_R e^{ik_1(-x-v_1t)+i\delta R}. \tag{11}$$

$$f_+ = A_T e^{ik_2(x-v_2t)+i\delta_T}. \tag{12}$$

In order that the boundary conditions be satisfied at all times, the frequencies on the two sides must be the same, so $k_1 v_1 = k_2 v_2 = \omega$. Setting $x = 0$, then, we have the equations:

$$A_I + A_R e^{i\delta_R} = A_T e^{i\delta_T}. \tag{13}$$

$$k_1 A_I - k_1 A_R e^{i\delta_R} = k_2 A_T e^{i\delta_T}. \tag{14}$$

Note $k_i = \omega_i / v_i, i = 1, 2$. Calling $\tilde{A}_R = A_R e^{i\delta R}$, etc., we have a simple system of linear equations to solve:

$$\tilde{A}_I + \tilde{A}_R = \tilde{A}_T \tag{15}$$

$$(\tilde{A}_I - \tilde{A}_R) = \frac{v_1}{v_2}\tilde{A}_T = \frac{v_1}{v_2}(\tilde{A}_I + \tilde{A}_R)$$

So

$$\tilde{A}_R = \frac{\tilde{A}_I(1 - v_1/v_2)}{1 + v_1/v_2} = \tilde{A}_I \left(\frac{v_1 - v_2}{v_1 + v_2}\right) \tag{16}$$

$$\tilde{A}_T = \frac{2v_2}{v_1 + v_2}\tilde{A}_I.$$

Or, putting back the δ's:

$$A_R e^{i\delta_R} = \frac{A_I(v_1 - v_2)}{v_1 + v_2} \qquad A_T e^{i\delta_T} = \frac{2v_2}{v_1 + v_2} A_I \tag{17}$$

We see that $\delta_T = 0$, while if $v_1 > v_2$, $\delta_R = 0$, and if $v_1 < v_2$, $\delta_R = \pi$. Remember:

$$v_i^2 = \frac{T_i}{\mu_i}. \tag{18}$$

Note the general strategy: we express the strength of the reflected and transmitted waves in terms of that of the incident wave. We also determine the phase relation. We will need to do the same thing for electromagnetic waves. This will be slightly more complicated due to the two possible ways to polarize the waves, but the ideas are identical. One enumerates the boundary conditions, and solves for the reflected and transmitted amplitudes in terms of the incoming amplitudes.

Wave Packets at Boundaries

Again, the description of boundaries in terms of plane waves is a bit troubling. For example, the incoming and reflected waves are *on top of each other*. But if we work with wave packets, the picture is more sensible. Before writing equations, let me just describe the result. Initially, one has a wave packing coming in from the left, with no reflected or transmitted wave. At some time (call it time $t = 0$), the wave reaches the boundary. At this point, the reflected and transmitted wave packets appear. Later, there is only a reflected wave packing, moving to the right, and a transmitted wave packet, moving to the left.

To see this mathematically, let's look at our earlier derivation of the motion of the wave packet in a different way. Start with

$$f(x,t) \propto \int dk \; e^{ik(x-vt)-\frac{(k-k_0)^2}{\Delta k^2}} \tag{19}$$

Now if $x - vt$ is much different than zero, the oscillations of the exponential factor tend to wash out the integral. One expects that the main contribution comes when the exponent, as a function of k, is most slowly varying, i.e. when

$$i(x-vt) = 2(k-k_0)/\Delta k^2. \tag{20}$$

The i is a bit weird, but we had this when we did the Gaussian integral earlier. Note also that we are assuming Δk is small, so k is near k_0 in the integral. We learn two things here. First, $x \approx vt$ wherever f is appreciable; second, if we just substitute back for $k - k_0$ in the exponent, we see that the result of the integral is approximately:

$$f \propto e^{ik(x-vt)} e^{-\frac{(x-vt)^2 \Delta k^2}{4}} \tag{21}$$

just as we found before.

But now take the case of the boundary. Now we have:

$$x < 0: \; f(x,t) \propto \int dk \; \left(A_I e^{ik(x-vt)-\frac{(k-k_0)^2}{\Delta k^2}} + A_R e^{ik(-x-vt)-\frac{(k-k_0)^2}{\Delta k^2}} \right) \tag{22}$$

$$x > 0: \; f(x,t) \propto \int dk \; A_T e^{ik(x-vt)-\frac{(k-k_0)^2}{\Delta k^2}} \tag{23}$$

But from our example above, we know how these integrals behave. The incoming term is appreciable when $x \approx vt, x < 0$, i.e. it is only appreciable for $t < 0$. The reflected term is only appreciable for $x \approx -vt, x < 0$, i.e. for $t > 0$! Similarly, the transmitted wave is only appreciable for $x \approx vt$, $x > 0$, i.e. it is only appreciable for $t > 0$. This is exactly the picture we outlined above.

Features of Electromagnetic Waves

From Maxwell's equations, with no charges or currents:

$$[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}]\vec{E} = 0 \quad [\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}]\vec{B} = 0 \tag{24}$$

with $c^2 = 1/(\epsilon_o \mu_o)$. These have plane wave solutions:

$$\vec{E}(\vec{r},t) = \vec{E}_o e^{i\vec{k}\cdot\vec{r}-i\omega t} \qquad \vec{B}(\vec{r},t) = \vec{B}_o e^{i\vec{k}\cdot\vec{r}-i\omega t} \tag{25}$$

where $\omega = c|\vec{k}|$. From Maxwell's equations,

$$\vec{\nabla}\cdot\vec{E} = 0 \Longrightarrow \vec{k}\cdot\vec{E}_o = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0 \Longrightarrow \vec{k}\cdot\vec{B}_o = 0 \tag{26}$$

and from

$$\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}\vec{B} \tag{27}$$

$$\vec{B} = \frac{\vec{k}\times\vec{E}_o}{\omega} = \frac{\hat{k}\times\vec{E}_o}{v} \tag{28}$$

The energy and momentum density of a plane wave can be worked out from the formulas for u and $\vec{S}$. It is useful to time-average. This gives:

$$\langle u\rangle = \frac{1}{2}\epsilon_o E_o^2 \quad \langle\vec{S}\rangle = \frac{1}{2}c\epsilon_o E_o^2\hat{k} \tag{29}$$

In matter, for linear media, the equations are very similar to those above, with c replaced by

$$v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{c}{n} \quad n = \sqrt{\frac{\epsilon\mu}{\epsilon_o\mu_o}} \tag{30}$$

ϵ, and correspondingly n, can be complex. This leads to dissipation. n is also, in general, a function of frequency, which leads to dispersion. A simple model of a material as a collection of harmonic oscillators, with characteristic frequency ω_o and damping constant γ gives for the real part of the index of refraction:

$$n = \frac{ck}{\omega} = 1 + \frac{Nq^2}{2m\epsilon_o}\frac{\omega_o^2-\omega^2}{(\omega_o^2-\omega^2)^2+\gamma^2\omega^2} \tag{31}$$

and for the damping part (twice the imaginary part:

$$\alpha = \frac{Nq^2\omega^2}{2m\epsilon_o}\frac{\gamma}{(\omega_o^2-\omega^2)^2+\gamma_2\omega^2}. \tag{32}$$

Boundary Conditions for Electromagnetic Waves

Boundary conditions are very important. From Maxwell's equations, one has:

$$\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp} \quad B_1^{\perp} = B_2^{\perp} \tag{33}$$

$$E_1^{\parallel} = E_2^{\parallel} \quad \frac{1}{\mu_1} B_1^{\parallel} = \frac{1}{\mu_2} B_2^{\parallel}. \tag{34}$$

So let's see what happens. The problem is a bit more complicated than the string case, because of the two possible polarizations of the waves (and the fact that we have to deal with both $\vec{E}$ and $\vec{B}$.

It is helpful to consider various special cases (refer to your book and to lecture for figures).

Normal Incidence:

$$E_{0I} + E_{0R} = E_{0T} \tag{35}$$

and

$$\frac{1}{\mu_1}\left(\frac{1}{v_1} E_{0I} - \frac{1}{v_1} E_{0R}\right) = \frac{1}{\mu_2}\frac{1}{v_2} E_{0T}. \tag{36}$$

After a little algebra:

$$E_{0R} = \left(\frac{1-\beta}{1+\beta}\right) E_{0I} \quad E_{0T} = \frac{2}{1+\beta} E_{0I} \quad \beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1} \tag{37}$$

Note the parallels to the string example. In the case that $\mu_i \approx \mu_0$ (typical), then

$$E_{0R} = \frac{v_2 - v_1}{v_2 + v_1} = \frac{\frac{1}{v_1} - \frac{1}{v_2}}{\frac{1}{v_1} + \frac{1}{v_2}} E_{0I} = \frac{n_1 - n_2}{n_1 + n_2} E_{0I} \tag{38}$$

Fraction of energy reflected, transmitted:

$$I = \frac{1}{2} v \epsilon E^2 \tag{39}$$

so

$$R = \frac{I_R}{I_I} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 \tag{40}$$

$$T = \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left(\frac{E_{0T}}{E_{0I}}\right)^2 = \frac{4 n_1 n_2}{(n_1 + n_2)^2}. \tag{41}$$

("reflection" and "transmission" coefficients). Note $T + R = 1$.

Oblique Incidence

This is the general case. First, there are several results which follow by virtue of the fact that there are boundary conditions to be satisfied. The equality of frequencies now means:

$$k_I v_1 = k_R v_1 = k_T v_2 \tag{42}$$

while the fact that the phase must be equal everywhere on the boundary translates into the statement (taking $\vec{k}_i$ in the xz plane):

$$k_{Ix} = k_{Rx} = k_{Tx} \tag{43}$$

This gives two important relations, independent of any details of the electric and magnetic fields:

$$\sin\theta_I = \sin\theta_R \tag{44}$$

and

$$\frac{\sin\theta_T}{\sin\theta_R} = \frac{n_1}{n_2} \tag{45}$$

The first of these states that the angle of incidence is the angle of reflection; second is Snell's law.

Now the boundary conditions on the fields give:

$$\epsilon_1(\vec{E}_{0I} + \vec{E}_{0R})_z = \epsilon_2(\vec{E}_{0T})_z \tag{46}$$

$$(\vec{B}_{0I} + \vec{B}_{0R})_z = (\vec{B}_{0T})_z$$

$$(\vec{E}_{0I} + \vec{E}_{0R})_{x,y} = (\vec{E}_{0T})_{x,y} \tag{47}$$

$$\frac{1}{\mu_1}(\vec{B}_{0I} + \vec{B}_{0R})_{x,y} = \frac{1}{\mu_2}(\vec{B}_{0T})_{x,y}$$

Now we break this into two cases:

1. Polarization Parallel to Plane of Incidence
2. Polarization Perpendicular to Plane of Incidence

We will focus on the first case. We can be more explicit about the boundary conditions: Perpendicular E (E_z):

$$\epsilon_I(-E_{0I}\sin\theta_I + E_{0R}\sin\theta_R) = \epsilon_2(-E_{0T}\sin\theta_T)$$

$\vec{B}$ is in the y direction, so there is no perp. condition. For the parallel components, we have:

$$E_{0I}\cos\theta_I + E_{0R}\cos\theta_R = E_{0T}\cos\theta_T.$$

and from the parallel component of B:

$$\frac{1}{\mu_1 v_1}(E_{0I} + E_{0R}) = \frac{1}{\mu_2 v_2}E_{0I}$$

This is actually the same as the $\vec{E}$ perp equation; to see this, divide by $\sin\theta_R$ and use Snell's law. calling

$$\alpha = \frac{\cos\theta_T}{\cos\theta_I} \qquad \beta = \frac{\mu_1 n_2}{\mu_2 n_1}$$

gives the *Fresnel equations*:

$$E_{0R} = \frac{\alpha - \beta}{\alpha + \beta}E_{0I}$$

$$E_{0T} = \frac{2}{\alpha + \beta}E_{0I}.$$

Perhaps not surprisingly, the amplitudes of the reflected and transmitted waves depend on the angle of incidence.

Special cases

1. $\theta_I = \frac{\pi}{2}$: (grazing incidence) α diverges. Total reflection.
2. Brewster's angle: $\alpha = \beta$; no reflected wave. Some algebra:

$$\sin^2\theta_I = \frac{1-\beta^2}{\left(\frac{n_1}{n_2}\right)^2 - \beta^2}$$

For conductors, $\omega_o = 0$. A good conductor will be characterized by a small γ, so

$$\epsilon \approx \frac{Nq^2}{m\epsilon_o} \frac{i}{\gamma\omega}. \tag{48}$$

and

$$k = \frac{\omega}{v} \approx \omega\sqrt{\epsilon} = e^{\frac{i\pi}{4}} \sqrt{\frac{nq^2}{m\epsilon_o\gamma}} \tag{49}$$

We can also derive this relation thinking about $\vec{J} = \sigma E$,

$$k \approx e^{\frac{i\pi}{4}} \sqrt{\frac{\omega\sigma}{2}}. \tag{50}$$

This gives a relation between σ and the damping term, which we understood more microscopically in terms of the mean free path of electrons in the material.

Dispersion: When the speed of light is not constant as a function of frequency, one encounters the phenomenon of dispersion. The basic idea is that $\omega = \omega(k)$, where $\omega(k)$ is more complicated than ck. So consider a Gaussian wave packet (for simplicity in one dimension), i.e.

$$f(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} g(k) e^{ikx - i\omega(k)t}. \tag{51}$$

Take

$$g(k) = Ce^{-(k-k_o)^2/(\Delta k)^2}. \tag{52}$$

This solves the wave equation, with a variable speed of light (check!). This integral is hard for a general function, but if Δk is small, than we can expand:

$$\omega(k) = \omega(k_o) + (k - k_o)v_g \qquad v_g = \frac{d\omega}{dk}|_{k_o}. \tag{53}$$

The resulting integral is a standard gaussian integral, which is done by completing the squares

$$\int_{-\infty}^{\infty} du e^{-au^2+bu} = \sqrt{\frac{\pi}{a}} e^{b^2/4} \tag{54}$$

and the result is:

$$f(x,t) \approx e^{i(k_o x - \omega_o t)} e^{-(x-v_g t)^2(\Delta k)^2/4}. \tag{55}$$

which is a plane wave modulated by a Gaussian. If we were a bit more careful, keeping the next term in the Taylor expansion of ω, we would see that the wave packet spreads in time. We now have

$$\omega(k) = \omega(k_o) + (k - k_o)v_g + \frac{1}{2}\nu(k - k_o)^2. \tag{56}$$

We still have to do a Gaussian integral, and can use the same formula as before, though the algebra is a bit messier. The result for $f(x,t)$ is still a Gaussian, but now proportional to

$$\exp\left[-(x - v_g t)^2(\Delta k)^2 \frac{1}{[1 + \nu(\Delta k)^2 t^2/4]}(1 - i/2\nu(\Delta k)^2 t)\right]. \tag{57}$$

The prefactor now depends on t as well. But note that the width increases with time; there is now also a more complicated phase factor.

Waveguides

Here we show that for TM and TE modes, one can obtain the transverse components of the fields in terms of E_z or B_z, respectively.

From Faraday's law we have:

$$\partial_x E_y - \partial_y E_x = i\omega B_z \tag{58}$$

$$ikE_x - \partial_x E_z = i\omega B_y$$

$$\partial_y E_z - ikE_y = i\omega B_x$$

Similarly, from Ampere

$$\partial_x B_y - \partial_y B_x = -i\frac{\omega}{c^2}E_z \tag{59}$$

$$ikB_x - \partial_x B_z = -i\frac{\omega}{c^2}B_y$$

$$\partial_y B_z - ikB_y = i\omega B_x$$

These equations can be solved for E_x, E_y, B_x, B_y in terms of $\partial_i E_z$, $\partial_i B_z$. E.g. in second and third equations, solve for B_y, B_x and substitute in last two equations. This gives equations for E_x, E_y, with solutions:

$$E_x = \frac{i}{(\omega/c)^2 - k^2}\left(k\partial_x E_z + \omega\partial_y B_z\right). \tag{60}$$

$$E_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\partial_y E_z - \omega\partial_x B_z\right).$$

There are similar solutions for B_x, B_y:

$$B_x = \frac{i}{(\omega/c)^2 - k^2}\left(k\partial_x B_z - \omega/c^2\partial_y E_z\right). \tag{61}$$

$$B_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\partial_y B_z + \omega/c^2\partial_x E_z\right).$$

So if we can solve for E_z, B_z, we know all of the fields.