Physics 110B. Electricity and Magnetism. Professor Dine

Spring 2006: FINAL EXAM. Take Home. Due Monday, June 10. 4:00 PM (Physics Office)

Problem 1. For an electromagnetic wave of the form:

$$\vec{E} = \vec{E}_o e^{ikz - i\omega t}, \tag{1}$$

use Maxwell's equations to: a. show that the polarization of the wave is perpendicular to its direction of motion.
Solution: From $\vec{\nabla} \cdot \vec{E} = 0$, $\hat{z} \cdot \vec{E}_o = 0$.
b. find the relation between ω and k, and from this the velocity of the wave (phase velocity).
Solution: This follows from the wave equation. Plugging in, gives the equation $\omega^2 = c^2 k^2$.
c. Find the relation between $\vec{E}$ and $\vec{B}$.

Solution:
This you can find from

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}.$$

For our solution, this reads:

$$ik\hat{z} \times \vec{E}_o = i\omega \vec{B}_o$$

so

$$\vec{B}_o = \frac{k}{\omega} \vec{k} \times \vec{E}_o = \frac{1}{c} \hat{k} \times \vec{E}_o$$

Problem 2. A particle moves in a circle in the $x - y$ plane.
a. Write a formula for the trajectory of the particle, assuming that the radius of the circle is ρ and the frequency of rotation is ω.
Solution: A suitable trajectory is:

$$\vec{x}(t) = \rho(\hat{x} \cos(\omega t) + \hat{y} \sin(\omega t))$$

b. Compute the dipole moment of the particle, relative to the origin.

Solution:

$$\vec{p}(t) = q\vec{x}(t)$$

Note that

$$\frac{d^2}{dt^2} \vec{p} = -\omega^2 \vec{p}.$$

c. Compute the $\vec{E}$ and $\vec{B}$ fields far away from the formula for dipole radiation.

Solution: Here you can plug in to eqn. 11.56, 11.57 of your text,

$$\vec{E} = -\frac{\mu_o \omega^2}{4\pi r} [\hat{r} \times (\hat{r} \times \frac{d^2 \vec{p}}{dt^2})]$$

and

$$\vec{B} = \frac{\mu_o \omega^2}{4\pi r c} [\hat{r} \times \frac{d^2 \vec{p}}{dt^2}].$$

These expressions can be simplified by using the triple product identity:

$$\vec{E} = -\frac{\mu_o \omega^2}{4\pi r}[\hat{r}\left(\hat{r}\cdot\frac{d^2\vec{p}}{dt^2}\right) - \frac{d^2\vec{p}}{dt^2}].$$

Taking $\frac{d^2\vec{p}}{dt^2}$ along the z axis, the dot product above is just $\cos(\theta)$.
d. Compute the $\vec{E}$ and $\vec{B}$ fields from the formulas we derived for the $\vec{E}$ and $\vec{B}$ fields of a charged particle. Remember, radiation requires acceleration.

Solution: Look at eqn. 10.65. We are only interested in the acceleration term. Also, our derivation of the dipole formula required non-relativistic velocities, so we can make the same approximation here. For low velocities, $\vec{u} \approx c\hat{r}$, so

$$\vec{E} \approx \frac{1}{4\pi\epsilon_o c^2 r}[\hat{r}\times(\hat{r}\times\frac{d^2\vec{x}}{dt^2})].$$

But $\frac{d^2\vec{x}}{dt^2} = -\omega^2\vec{x}$, for our circular trajectory. Noting the definition of $\vec{p}$, and $c^2 = 1/(\mu_o\epsilon_o)$, gives our formula for $\vec{E}$. $\vec{B}$ works similarly.

Problem 3. Work out the following in terms of ordinary three vectors and scalars: a. $x^\mu k_\mu$. If $k = (\omega/c, \vec{k})$, show that $e^{ix^\mu k_\mu}$ is the usual expression for a plane wave.

Solution:

$$x^\mu k_\mu = -x^o k^o + \vec{k}\cdot\vec{x} = -\omega t + \vec{k}\cdot\vec{x}.$$

This is just the phase of our standard plane wave.
b. Show that $p^\mu p_\mu$ is $-m^2c^2$, for a particle of mass m.

Solution:

$$p^\mu p_\mu = -E^2/c^2 + \vec{p}^2 = \frac{1}{c^2}(m^2c^4 + \vec{p}^2c^2) - \vec{p}^2 = -m^2c^2.$$

Problem 4. A relativistic wave equation: The infamous Higgs boson obeys a wave equation similar to that for light. It is simpler, since there is just one field, ϕ. The equation is:

$$[-\vec{\nabla}^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \omega_o^2/c^2]\phi = 0. \tag{2}$$

Here ω_o is a constant.
a. Show that $e^{i(\vec{k}\cdot\vec{x}-\omega t)}$ solves the wave equation. What is the relation between ω and k?

Solution: Plugging in the equation, we obtain:

$$c^2k^2 - \omega^2 + \omega_o^2$$

or

$$\omega^2 = c^2k^2 + \omega_o^2.$$

b. What is the group velocity of the waves?

Solution:

$$v_g = \frac{\partial\omega}{\partial k} = \frac{ck}{\omega}$$

c. Now let's do some quantum mechanics and relativity together. If $p = \hbar k$, and $E = \hbar\omega$, what is $p^\mu p_\mu$? Interpret this in terms of the mass of the Higgs boson.

Solution: Using these relations, we have:

$$E^2/c^2 - p^2 = \hbar^2\omega_o^2/c^2 = -m^2c^2$$

, or

$$m^2 = \frac{\hbar^2\omega_o^2}{c^4}$$

d. Given your result in part b, does the particle ever move faster than the speed of light?

Solution: No, since $v_g < c$ if $\omega_o > 0$.

Problem 5. An electromagnetic wave is described, in the gauge $\vec{\nabla} \cdot \vec{A} = 0$ by a vector potential of the form:

$$\vec{A}(\vec{x}, t) = \vec{A}_o e^{i(\vec{k}\cdot\vec{x} - \omega t)}. \tag{3}$$

The scalar potential vanishes, $V = 0$.
a. What does the gauge condition imply about $\vec{A}_o$?

Solution:

$$\vec{k} \cdot \vec{A}_o = 0$$

i.e. the vector potential is transverse to the direction of motion of the wave.
b. Calculate the $\vec{E}$ and $\vec{B}$ fields. Check that they are transverse. Show that they obey the usual relations,

$$\vec{B} = \frac{1}{c}\hat{k} \times \vec{E}. \tag{4}$$

Solution: Both are simple, especially since $V = 0$.

$$\vec{E} = \frac{1}{c}\frac{\partial \vec{A}}{\partial t} = -i\omega\vec{A}_o e^{i(\vec{k}\cdot\vec{x} - \omega t)}.$$

The i here means that $\vec{A}$ and $\vec{E}$ are 90^o out of phase. Similarly:

$$\vec{B} = \vec{\nabla} \times \vec{A} = i\vec{k} \times \vec{A}_o e^{i(\vec{k}\cdot\vec{x} - \omega t)}.$$

Again note the i, which indicates the phase relation.
c. Express the energy density and the Poynting vector in terms of $\vec{A}$.

Solution: It is helpful, first, to remember that time averaging,

$$< \text{Re } e^{-i\omega t} \text{Re } e^{-i\omega t} >= \frac{1}{2} \qquad < \text{Im } e^{-i\omega t} \text{Im} >= \frac{1}{2}.$$

The factors of i pull out the imaginary part of the exponential. So

$$u_{em} = \frac{1}{2}(\epsilon_o E^2 + \frac{1}{\mu_o}B^2) = \frac{1}{4}\left(\epsilon_o/c^2\omega^2\vec{A}_o^2 + \frac{1}{\mu_o}(\vec{k} \times \vec{A}_o)^2\right).$$

But $(\vec{k} \times \vec{A}_o)^2 = k^2A_o^2 - (\vec{k} \cdot \vec{A}_o)^2 = k^2A_o^2$, so the whole expression simplifies to:

$$u_{em} = \frac{1}{2}k^2\epsilon_o A_o^2.$$

Similarly,

$$\vec{S} = \mu_o\vec{E} \times \vec{B} = \frac{\mu_o}{2c}\omega\vec{A}_o \times (\vec{k} \times \vec{A}_o$$

Using the triple product identity and the transversality of $\vec{A}_o$, gives

$$\vec{S} = \frac{\mu_o}{2}k^2A_o^2\hat{k}.$$

Problem 6. Derive a formula for ϵ for a conductor, assuming that the electrons in the conductor are free (there is no harmonic restoring force). Show that the real and imaginary parts of ϵ are equal in magnitude. Write a formula for the attenuation of a wave in the conductor with distance.

Solution: This we did in class. Start with the equation of motion:

$$\ddot{x} + \gamma \dot{x} = \frac{qE}{m}$$

with

$$E = E_o e^{-i\omega t}$$

describing the sinusoidal behavior of the field at the position of the particle. The solution is:

$$x = \frac{qE_o}{-im\gamma} e^{-i\omega t}$$

To get the susceptibility, we multiply this by q and the density. At low frequencies, because of the ω, this gives:

$$\epsilon \approx \frac{Nq^2}{m\epsilon_o \gamma\gamma} i$$

(compare eqn. 9.161 of your text). Now

$$k = \frac{\omega}{c}\sqrt{\epsilon}$$

as usual, so, from

$$\sqrt{i} = e^{i\pi/4} = \frac{1+i}{\sqrt{2}}$$

$$k = \alpha(1+i) \quad \alpha = \frac{1}{2}\sqrt{\frac{Nq^2\omega}{m\epsilon_o\gamma}}$$

and $1/\alpha$ is the attenuation length.