Problem 1. For an electromagnetic wave of the form:
\[ \vec{E} = \vec{E}_0 e^{ikz - i\omega t}, \]  

use Maxwell’s equations to:  

a. show that the polarization of the wave is perpendicular to its direction of motion.  

**Solution:** From \( \vec{\nabla} \cdot \vec{E} = 0, \vec{z} \cdot \vec{E}_0 = 0 \).

b. find the relation between \( \omega \) and \( k \), and from this the velocity of the wave (phase velocity).  

**Solution:** This follows from the wave equation. Plugging in, gives the equation \( \omega^2 = c^2 k^2 \).

c. Find the relation between \( \vec{E} \) and \( \vec{B} \).

**Solution:**  

This you can find from  
\[ \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}. \]

For our solution, this reads:
\[ ik \hat{z} \times \vec{E}_0 = i\omega \vec{B}_0 \]

so
\[ \vec{B}_0 = \frac{k}{\omega} \hat{k} \times \vec{E}_0 = \frac{1}{c} \hat{k} \times \vec{E}_0 \]

Problem 2. A particle moves in a circle in the \( x - y \) plane.

a. Write a formula for the trajectory of the particle, assuming that the radius of the circle is \( \rho \) and the frequency of rotation is \( \omega \).

**Solution:** A suitable trajectory is:
\[ \vec{x}(t) = \rho(\dot{x} \cos(\omega t) + \dot{y} \sin(\omega t)) \]

b. Compute the dipole moment of the particle, relative to the origin.

**Solution:**  
\[ \vec{p}(t) = q\vec{x}(t) \]

Note that
\[ \frac{d^2}{dt^2} \vec{p} = -\omega^2 \vec{p}. \]

c. Compute the \( \vec{E} \) and \( \vec{B} \) fields far away from the formula for dipole radiation.

**Solution:** Here you can plug in to eqn. 11.56, 11.57 of your text,
\[ \vec{E} = -\frac{\mu_0 \omega^2}{4\pi r}[\hat{r} \times (\hat{r} \times \frac{d^2 \vec{p}}{dt^2})] \]

and
\[ \vec{B} = \frac{\mu_0 \omega^2}{4\pi rc}[\hat{r} \times \frac{d^2 \vec{p}}{dt^2}]. \]
These expressions can be simplified by using the triple product identity:

\[
\vec{E} = -\frac{\mu_0\omega^2}{4\pi r} \left[ \hat{r} \cdot \left( \frac{d^2\vec{p}}{dt^2} \right) - \frac{d^2\vec{p}}{dt^2} \right].
\]

Taking \( \frac{d^2\vec{p}}{dt^2} \) along the \( z \) axis, the dot product above is just \( \cos(\theta) \).

d. Compute the \( \vec{E} \) and \( \vec{B} \) fields from the formulas we derived for the \( \vec{E} \) and \( \vec{B} \) fields of a charged particle. Remember, radiation requires acceleration.

**Solution:** Look at eqn. 10.65. We are only interested in the acceleration term. Also, our derivation of the dipole formula required non-relativistic velocities, so we can make the same approximation here. For low velocities, \( \vec{u} \approx c \hat{r} \), so

\[
\vec{E} \approx \frac{1}{4\pi\varepsilon_0 c^2 r} [\hat{r} \times (\hat{r} \times \frac{d^2\vec{r}}{dt^2})].
\]

But \( \frac{d^2\vec{r}}{dt^2} = -\omega^2\vec{x} \), for our circular trajectory. Noting the definition of \( \vec{p} \), and \( c^2 = 1/(\mu_0\varepsilon_0) \), gives our formula for \( \vec{E} \). \( \vec{B} \) works similarly.

**Problem 3.** Work out the following in terms of ordinary three vectors and scalars: a. \( x^\mu k_\mu \). If \( k = (\omega/c, \vec{k}) \), show that \( e^{ix^\mu k_\mu} \) is the usual expression for a plane wave.

**Solution:**

\[
x^\mu k_\mu = -x^0 k^0 + \vec{k} \cdot \vec{x} = -\omega t + \vec{k} \cdot \vec{x}.
\]

This is just the phase of our standard plane wave.

b. Show that \( p^\mu p_\mu \) is \(-m^2c^2\), for a particle of mass \( m \).

**Solution:**

\[
p^\mu p_\mu = -E^2/c^2 + \vec{p}^2 = \frac{1}{c^2}(m^2c^4 + \vec{p}^2c^2) - \vec{p}^2 = -m^2c^2.
\]

**Problem 4.** A relativistic wave equation: The infamous Higgs boson obeys a wave equation similar to that for light. It is simpler, since there is just one field, \( \phi \). The equation is:

\[
[-\vec{\nabla}^2 + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \omega_0^2/c^2] \phi = 0.
\] (2)

Here \( \omega_0 \) is a constant.

a. Show that \( e^{i(k\vec{x} - \omega t)} \) solves the wave equation. What is the relation between \( \omega \) and \( k \)?

**Solution:** Plugging in the equation, we obtain:

\[
c^2k^2 - \omega^2 + \omega_0^2
\]

or

\[
\omega^2 = c^2k^2 + \omega_0^2.
\]

b. What is the group velocity of the waves?

**Solution:**

\[
v_g = \frac{\partial \omega}{\partial k} = \frac{ck}{\omega}
\]

C. Now let’s do some quantum mechanics and relativity together. If \( p = \hbar k \), and \( E = \hbar \omega \), what is \( p^\mu p_\mu \)? Interpret this in terms of the mass of the Higgs boson.

**Solution:** Using these relations, we have:
\( E^2/c^2 - p^2 = h^2 \omega_o^2/c^2 = -m^2c^2 \)

, or

\[ m^2 = \frac{h^2 \omega_o^2}{c^4} \]

d. Given your result in part b, does the particle ever move faster than the speed of light?

**Solution:** No, since \( v_g < c \) if \( \omega_o > 0 \).

**Problem 5.** An electromagnetic wave is described, in the gauge \( \vec{\nabla} \cdot \vec{A} = 0 \) by a vector potential of the form:

\[ \vec{A}(\vec{x}, t) = \vec{A}_o e^{i(\vec{k} \cdot \vec{x} - \omega t)}. \]  

The scalar potential vanishes, \( V = 0 \).

a. What does the gauge condition imply about \( \vec{A}_o \)?

**Solution:**

\[ \vec{k} \cdot \vec{A}_o = 0 \]

i.e. the vector potential is transverse to the direction of motion of the wave.

b. Calculate the \( \vec{E} \) and \( \vec{B} \) fields. Check that they are transverse. Show that they obey the usual relations,

\[ \vec{B} = \frac{1}{c} \vec{k} \times \vec{E}. \]  

**Solution:** Both are simple, especially since \( V = 0 \).

\[ \vec{E} = \frac{1}{c} \frac{\partial \vec{A}}{\partial t} = -i \omega \vec{A}_o e^{i(\vec{k} \cdot \vec{x} - \omega t)}. \]

The \( i \) here means that \( \vec{A} \) and \( \vec{E} \) are 90° out of phase. Similarly:

\[ \vec{B} = \vec{\nabla} \times \vec{A} = i \vec{k} \times \vec{A}_o e^{i(\vec{k} \cdot \vec{x} - \omega t)}. \]

Again note the \( i \), which indicates the phase relation.

c. Express the energy density and the Poynting vector in terms of \( \vec{A} \).

**Solution:** It is helpful, first, to remember that time averaging,

\[ <\text{Re} e^{-i\omega t} \text{Re} e^{-i\omega t} > = \frac{1}{2} \quad <\text{Im} e^{-i\omega t} \text{Im} > = \frac{1}{2}. \]

The factors of \( i \) pull out the imaginary part of the exponential. So

\[ u_{em} = \frac{1}{2} (\epsilon_o E^2 + \frac{1}{\mu_o} B^2) = \frac{1}{4} \left( \epsilon_o/c^2 \omega^2 \vec{A}_o^2 + \frac{1}{\mu_o} (\vec{k} \times \vec{A}_o)^2 \right). \]

But \( (\vec{k} \times \vec{A}_o)^2 = k^2 A_o^2 - (\vec{k} \cdot \vec{A}_o)^2 = k^2 A_o^2 \), so the whole expression simplifies to:

\[ u_{em} = \frac{1}{2} k^2 \epsilon_o A_o^2. \]

Similarly,

\[ \vec{S} = \mu_o \vec{E} \times \vec{B} = \frac{\mu_o}{2c\omega} \vec{A}_o \times (\vec{k} \times \vec{A}_o) \]

Using the triple product identity and the transversality of \( \vec{A}_o \), gives

\[ \vec{S} = \frac{\mu_o}{2} k^2 A_o^2 \vec{k}. \]
Problem 6. Derive a formula for $\epsilon$ for a conductor, assuming that the electrons in the conductor are free (there is no harmonic restoring force). Show that the real and imaginary parts of $\epsilon$ are equal in magnitude. Write a formula for the attenuation of a wave in the conductor with distance.

Solution: This we did in class. Start with the equation of motion:

$$\ddot{x} + \gamma \dot{x} = \frac{qE}{m}$$

with

$$E = E_0 e^{-i\omega t}$$
describing the sinusoidal behavior of the field at the position of the particle. The solution is:

$$x = \frac{qE_0}{-im\gamma} e^{-i\omega t}$$

To get the susceptibility, we multiply this by $q$ and the density. At low frequencies, because of the $\omega$, this gives:

$$\epsilon \approx \frac{Nq^2}{m\epsilon_0 \gamma^2} i$$

(compare eqn. 9.161 of your text). Now

$$k = \frac{\omega}{c} \sqrt{\epsilon}$$
as usual, so, from

$$\sqrt{i} = e^{i\pi/4} = \frac{1 + i}{\sqrt{2}}$$

$$k = \alpha (1 + i) \quad \alpha = \frac{1}{2} \sqrt{\frac{Nq^2 \omega}{m\epsilon_0 \gamma}}$$

and $1/\alpha$ is the attenuation length.