1 The Potential of a Moving Point Particle

Starting, for example, with

$$V(\vec{x}, t) = \int d^3 \vec{x}' dt' \frac{1}{|\vec{x} - \vec{x}'|} \rho(\vec{x}', t') \delta(t - t' - \frac{1}{c}|\vec{x} - \vec{x}'|).$$

(1)

$$\vec{A}(\vec{x}, t) = \int d^3 \vec{x}' dt' \frac{1}{|\vec{x} - \vec{x}'|} \vec{J}(\vec{x}', t') \delta(t - t' - \frac{1}{c}|\vec{x} - \vec{x}'|).$$

(2)

one might think, for a point particle, one would simply obtain, for $V$, say,

$$V(\vec{x}, t) = \frac{1}{4\pi \epsilon_0} \frac{1}{|\vec{x} - \vec{x}_0(t)|}.$$  

(3)

but this is not correct. To understand this, we can work more carefully (as we will below) with the $\delta$-function. But we can see the issue by using our representation of the $\delta$-function as a very narrow Gaussian (where we take the limit in the end). In other words, we write

$$\rho(\vec{x}, t) = \left(\frac{1}{\sqrt{\pi} \sigma}\right)^3 e^{-\frac{|\vec{x} - \vec{x}_0(t)|^2}{\sigma^2}}.$$  

(4)

(and similarly for $\vec{J}$). When we do the integrals above, we need to be careful about the fact that the retarded time depends on $\vec{x}'$. Let’s look at this carefully. Consider $V(\vec{c}, t)$:

$$\frac{q}{4\pi \epsilon_0} \int d^3 \vec{x}' \frac{1}{|\vec{c} - \vec{x}'|} \frac{1}{(\sqrt{\pi} \sigma)^3} e^{-\frac{|\vec{x}' - \vec{x}_0(t)|^2}{\sigma^2}}.$$  

(5)

To do the integral, we note that for small $\sigma$, $\vec{x}' \approx \vec{x}_0$, so we write:

$$\vec{x}' = \vec{x}_0(t_R) + \vec{u}.$$  

(6)

where

$$\vec{t}_R = t - \frac{|\vec{x} - \vec{x}_0(t_R)|}{c}.$$  

(7)

Now, we can write:

$$t_r = t - \frac{|\vec{x} - \vec{x}'|}{c} = t - \frac{|\vec{x} - \vec{x}_0(t_R) - \vec{u}|}{c}.$$  

$$= t - \frac{|\vec{x} - \vec{x}_0(t_R)|}{c} + \frac{\vec{u} \cdot (\vec{x} - \vec{x}_0(t_R))}{c|\vec{x} - \vec{x}_0(t_R)|}.$$  


\[ t_R + \frac{\vec{u} \cdot \vec{R}}{cR} \]

where

\[ \vec{R} = \vec{x} - \vec{x}_0(t_R). \]  

(9)

Using this result, we can write the objection in the exponential as:

\[ |\vec{x}' - \vec{x}_0(t_r)|^2 = |\vec{x}' - \vec{x}_0(t_R) - \vec{v}(t_r - t_R)|^2 \]

(10)

\[ = |\vec{u} - \vec{v} \frac{\vec{u} \cdot \vec{R}}{R}|^2. \]

So, finally, the integral is simple. Take \( \vec{v} \) along the \( x \) axis. Then the factor in the exponent becomes:

\[ u_y^2 + u_z^2 + u_x \left( 1 - \frac{vR_x}{R} \right)^2 \]

(11)

The integrals along the \( y \) and \( z \) directions just give \( \sqrt{\pi \sigma^2} \). The integral along the \( x \) direction gives an extra factor of

\[ \frac{1}{(1 - v \frac{R_x}{R})} \]

For a general direction (not \( x \)), the factor is:

\[ \frac{1}{1 - \vec{v} \cdot \frac{\vec{R}}{R}} \]

So from this we obtain:

\[ V(\vec{r}, t) = \frac{q}{4\pi\varepsilon_0} \frac{1}{R - \frac{1}{c} \vec{v}_0 \cdot \vec{R}} \]

(12)

\[ A(\vec{r}, t) = \mu_0 \frac{q \vec{v}}{4\pi} \frac{1}{R - \frac{1}{c} \vec{v}_0 \cdot \vec{R}} \]

(13)
2 The Lienard-Wiechert Potentials Directly from the Delta-Function

We can derive the scalar and vector potential for a point charge starting with the expressions we wrote for the scalar and vector potentials,

\[ V(\vec{x}, t) = \int d^3x' dt' \frac{1}{|\vec{x} - \vec{x}'|} \rho(\vec{x}', t') \delta(t - t' - \frac{1}{c} |\vec{x} - \vec{x}'|). \] (14)

\[ \vec{A}(\vec{x}, t) = \int d^3x' dt' \frac{1}{|\vec{x} - \vec{x}'|} \vec{J}(\vec{x}', t') \delta(t - t' - \frac{1}{c} |\vec{x} - \vec{x}'|). \] (15)

and the charge and current distributions we wrote for point charges:

\[ \rho(\vec{x}, t) = \sum_i q \delta(\vec{x} - \vec{x}_o(t)) \] 
\[ \vec{j}(\vec{x}, t) = \sum_i q \vec{v}_o(t) \delta(\vec{x} - \vec{x}_o(t)) \] (16)

where \( \vec{x}_o(t) \) is the position of the particle at time \( t \), and \( \vec{v}_o \) is its velocity.

We just need to figure out how to do the integral over the \( \delta \)-function. For a \( \delta \)-function, the most we care about is its behavior near the point where its argument vanishes. We called \( t_R \) the solution to this condition,

\[ t_R = t - \frac{1}{c} |\vec{x} - \vec{x}_o(t_R)|. \] (17)

What is somewhat complicated about this equation is that it is an implicit equation for \( t_R \). We can solve it, however, once we know the trajectories of the charged particle. At time \( t' = t_R + (t' - t_R) \) near \( t_R \), we can Taylor expand the position:

\[ \vec{x}_o(t) \approx \vec{x}_o(t_R) + (t' - t_R) \vec{v}_o(t_R) \] (18)

Using this, we can write:

\[ |\vec{x} - \vec{x}_o(t')| \approx |\vec{x} - \vec{x}_o(t_R) - (t' - t_R) \vec{v}_o(t_R)| \] (19)

Call \( \vec{R} = \vec{x} - \vec{x}_o(t_R) \); then

\[ |\vec{x} - \vec{x}_o(t')| \approx (\vec{R}^2 - 2 \vec{R} \cdot \vec{v}_o(t' - t_R))^{1/2} \] 
\[ \approx \vec{R} - \frac{\vec{R} \cdot \vec{v}_o}{\vec{R}} (t' - t_R) \] (20)

So finally, the argument of the \( \delta \)-function is:

\[ \delta([t - \frac{1}{c} \vec{R} - \frac{1}{c} \vec{v}_o \cdot \vec{R}] - t' (1 - \frac{1}{c} \vec{v}_o \cdot \vec{R})). \] (21)

Remember that \( t' \) is the integration variable and note that \( t' \) appears only in the second set of terms. The \( \delta \) function still vanishes when \( t' = t_R \). But what we also need is that:

\[ \delta(a(t' - t_R)) = \frac{1}{a} \delta(t' - t_R)). \] (22)

So from this we obtain:

\[ V(\vec{r}, t) = \frac{q}{4 \pi \epsilon_o} \frac{1}{\vec{R} - \frac{1}{c} \vec{v}_o \cdot \vec{R}} \] (23)

\[ \vec{A}(\vec{r}, t) = \mu_o \frac{q\vec{v}}{4 \pi} \frac{1}{\vec{R} - \frac{1}{c} \vec{v}_o \cdot \vec{R}} \] (24)

where in each case, the quantities on the right hand side are evaluated at the retarded time.
3 Evaluating the Fields

Our index notation is particularly effective in evaluating the $\vec{E}$ and $\vec{B}$ fields of a point charge. We need to evaluate:

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} V \quad \vec{B} = \vec{\nabla} \times \vec{A}. \quad (25)$$

We need to be careful, however, because $t_R$ is implicitly a function of $\vec{x}$. So when we take derivatives with respect to $\vec{x}$, we need to differentiate not only the terms with explicit $\vec{x}$'s, but also the terms with $t_R$. So we start by working out these derivatives. Differentiating both sides of:

$$t_R = t - \frac{1}{c} |\vec{x} - \vec{x}_o(t_R)| \quad (26)$$

remembering that

$$|\vec{x} - \vec{x}_o(t_R)| = ((x_i - x_{ai})^2)^{1/2} \quad (27)$$

gives

$$\partial_i t_R = -\frac{1}{c} \frac{R_i}{R} + \frac{\vec{v}_o(t_R)}{c} \cdot \frac{\vec{R}}{R} \partial_i t_R \quad (28)$$

Solving for $\partial_i t_R$:

$$\partial_i t_R = -\frac{R_i}{c R} \left( \frac{1}{1 - \frac{\vec{v}_o(t_R)}{c} \cdot \frac{\vec{R}}{R}} \right) \quad (29)$$

It will also be useful to have a formula for $\partial_i \mathcal{R}$. From

$$\mathcal{R} = c(t - t_R) \quad (30)$$

we have

$$\partial_i \mathcal{R} = -c \partial_i t_R. \quad (31)$$

So now we can start taking derivatives.

$$\partial_i V = -\frac{qc}{4\pi \epsilon_0 (Rc - \vec{R} \cdot \vec{v})^2} \partial_i (Rc - \vec{R} \cdot \vec{v}) \quad (32)$$

Now

$$\partial_i \vec{R} \cdot \vec{v} = \partial_i (r_j - x_{aj}(t_R)) \dot{x}_{aj}(t_R)$$

$$= \dot{x}_{oi} - \dot{x}_{oj} \partial_i t_R - \mathcal{R}_j \dot{x}_{oj} \partial_i t_R \quad (33)$$

So

$$\partial_i V = -\frac{qc}{4\pi \epsilon_0 (Rc - \vec{R} \cdot \vec{v})^2} \left( -c \partial_i t_R + v^2 \partial_i t_R + \vec{R} \cdot \vec{a} \partial_i t_R - v_i \right) \quad (34)$$

Using our expression for $\partial_i t_R$ gives:

$$\partial_i V = -\frac{qc}{4\pi \epsilon_0 (Rc - \vec{R} \cdot \vec{v})^2} \left[ -c^2 \mathcal{R}_i + v^2 \mathcal{R}_i + \vec{R} \cdot \vec{a} \mathcal{R}_i - v_i (\vec{R} \cdot \vec{v} - c\mathcal{R}) \right] \quad (35)$$

With a bit more algebra, one can show:

$$\frac{\partial \vec{A}}{\partial t} = \frac{1}{4\pi \epsilon_0} \frac{qc}{(Rc - \vec{R} \cdot \vec{v})^3} \left[ (Rc - \vec{R} \cdot \vec{v})(-\vec{v} + \mathcal{R} \vec{a}/c) + \frac{\mathcal{R}}{c} (c^2 - v^2 + \vec{R} \cdot \vec{a}) \vec{v} \right] \quad (36)$$

and combining these, you obtain:

$$\vec{E}(\vec{r}, t) = \frac{1}{4\pi \epsilon_0} \frac{\mathcal{R}}{(\vec{R} \cdot \vec{u})^3} \left[ (c^2 - v^2) \vec{u} + \vec{R} \times \vec{u} \times \vec{a} \right] \quad (37)$$
where \( \vec{u} = c\hat{\mathbf{R}} - \vec{v} \). Similarly,
\[
\vec{B} = \frac{1}{c} \hat{\mathbf{R}} \times \vec{E}(\vec{r}, t).
\] (38)

**Exercise:** Fill in the details of the calculations of \( \vec{E} \) and \( \vec{B} \), using the index notation as above.

Where does the energy go?
\[
\vec{S} \cdot \hat{\mathbf{r}} = \frac{1}{\mu_0} (\vec{E} \times \vec{B}) \cdot \hat{n}
\] (39)
\[
= \frac{q^2}{4\pi c R^2} \left| \frac{\hat{\mathbf{r}} \times (\hat{\mathbf{r}} - \hat{\beta}) \times \frac{q\hat{\beta}}{\mu} \left[ 1 - (1 - \hat{\beta} \cdot \hat{n})^3 \right]}{(1 - \hat{\beta} \cdot \hat{n})^3} \right|^2
\]

Note different behaviors if velocity parallel, perpendicular to acceleration (circular vs. linear motion). Also peaking with angle.