Physics 110. Electricity and Magnetism. Professor Dine

Spring, 2008. Handout: Alternative Approach to the Lienard-Wiechart Potentials

1 The Potential of a Moving Point Particle

Starting, for example, with

$$V(\vec{x},t) = \int d^3x'dt' \frac{1}{|\vec{x}-\vec{x}'|}\rho(\vec{x}',t')\delta(t-t'-\frac{1}{c}|\vec{x}-\vec{x}'|). \tag{1}$$

$$\vec{A}(\vec{x},t) = \int d^3x'dt' \frac{1}{|\vec{x}-\vec{x}'|}\vec{J}(\vec{x}',t')\delta(t-t'-\frac{1}{c}|\vec{x}-\vec{x}'|). \tag{2}$$

one might think, for a point particle, one would simply obtain, for V, say,

$$V(\vec{x},t) = \frac{1}{4\pi\epsilon_0}\frac{1}{|\vec{x}-\vec{x}_0(t)|} \tag{3}$$

but this is not correct. To understand this, we can work more carefully (as we will below) with the δ-function. But we can see the issue by using our representation of the δ-function as a very narrow Gaussian (where we take the limit in the end). In other words, we write

$$\rho(\vec{x},t) = \left(\frac{1}{\sqrt{\pi}\sigma}\right)^3 e^{-\frac{|\vec{x}-\vec{x}_0(t)|^2}{\sigma^2}}. \tag{4}$$

(and similarly for $\vec{J}$). When we do the integrals above, we need to be careful about the fact that the retarded time depends on $\vec{x}'$. Let's look at this carefully. Consider $V(\vec{c},t)$:

$$\frac{q}{4\pi\epsilon_0}\int d^3x' \frac{1}{|\vec{x}-\vec{x}'|}\frac{1}{(\sqrt{\pi\sigma^2})^3}e^{-\frac{|\vec{x}'-\vec{x}_0(t)|^2}{\sigma^2}}. \tag{5}$$

To do the integral, we note that for small σ, $\vec{x}' \approx \vec{x}_0$, so we write:

$$\vec{x}' = \vec{x}_0(t_R) + \vec{u} \tag{6}$$

where

$$\vec{t}_R = t - \frac{|\vec{x}-\vec{x}_0(t_R)|}{c} \tag{7}$$

Now, we can write:

$$\begin{aligned} t_r &= t - \frac{|\vec{x}-\vec{x}'|}{c} \\ &= t - \frac{|\vec{x}-\vec{x}_0(t_R)-\vec{u}|}{c} \\ &= t - \frac{|\vec{x}-\vec{x}_0(t_R)|}{c} + \frac{\vec{u}\cdot(\vec{x}-\vec{x}_0(t_R))}{c|\vec{x}-\vec{x}_0(t_R)|} \end{aligned} \tag{8}$$

$$= t_R + \frac{\vec{u} \cdot \vec{\mathcal{R}}}{c\mathcal{R}}$$

where

$$\vec{\mathcal{R}} = \vec{x} - \vec{x}_0(t_R). \tag{9}$$

Using this result, we can write the objection in the exponential as:

$$|\vec{x}' - \vec{x}_0(t_r)|^2 = |\vec{x}' - \vec{x}_0(t_R) - \vec{v}(t_r - t_R)|^2 \tag{10}$$

$$= |\vec{u} - \vec{v}\frac{\vec{u} \cdot \vec{\mathcal{R}}}{\mathcal{R}}|^2.$$

So, finally, the integral is simple. Take $\vec{v}$ along the x axis. Then the factor in the exponent becomes:

$$u_y^2 + u_z^2 + u_x \left(1 - \frac{v\mathcal{R}_x}{\mathcal{R}}\right)^2 \tag{11}$$

The integrals along the y and z directions just give $\sqrt{\pi\sigma^2}$. The integral along the x direction gives an extra factor of

$$\frac{1}{(1 - v\frac{\mathcal{R}_x}{\mathcal{R}})}$$

For a general direction (not x), the factor is:

$$\frac{1}{1 - \vec{v} \cdot \frac{\vec{\mathcal{R}}}{\mathcal{R}}}$$

So from this we obtain:

$$V(\vec{r}, t) = \frac{q}{4\pi\epsilon_o}\frac{1}{\mathcal{R} - \frac{1}{c}\vec{v}_o \cdot \vec{\mathcal{R}}} \tag{12}$$

$$\vec{A}(\vec{r}, t) = \mu_o \frac{q\vec{v}}{4\pi}\frac{1}{\mathcal{R} - \frac{1}{c}\vec{v}_o \cdot \vec{\mathcal{R}}} \tag{13}$$

2 The Lienard-Wiechart Potentials Directly from the Delta-Function

We can derive the scalar and vector potential for a point charge starting with the expressions we wrote for the scalar and vector potentials,

$$V(\vec{x},t) = \int d^3x'dt' \frac{1}{|\vec{x}-\vec{x}'|}\rho(\vec{x}',t')\delta(t-t'-\frac{1}{c}|\vec{x}-\vec{x}'|). \tag{14}$$

$$\vec{A}(\vec{x},t) = \int d^3x'dt' \frac{1}{|\vec{x}-\vec{x}'|}\vec{J}(\vec{x}',t')\delta(t-t'-\frac{1}{c}|\vec{x}-\vec{x}'|). \tag{15}$$

and the charge and current distributions we wrote for point charges:

$$\rho(\vec{x},t) = \sum_i q\delta(\vec{x}-\vec{x}_o(t)) \qquad \vec{j}(\vec{x},t) = \sum_i q\vec{v}_o(t)\delta(\vec{x}-\vec{x}_o(t)) \tag{16}$$

where $\vec{x}_o(t)$ is the position of the particle at time t, and $\vec{v}_o$ is its velocity.

We just need to figure out how to do the integral over the δ-function. For a δ-function, the most we care about is its behavior *near* the point where its argument vanishes. We called t_R the solution to this condition,

$$t_R = t - \frac{1}{c}|\vec{x}-\vec{x}_o(t_R)|. \tag{17}$$

What is somewhat complicated about this equation is that it is an implicit equation for t_R. We can solve it, however, once we know the trajectories of the charged particle. At time $t' = t_R + (t'-t_R)$ near t_R, we can Taylor expand the position:

$$\vec{x}_o(t) \approx \vec{x}_o(t_R) + (t'-t_R)\vec{v}_o(t_R) \tag{18}$$

Using this, we can write:

$$|\vec{x}-\vec{x}_o(t')| \approx |\vec{x}-\vec{x}_o(t_R) - (t'-t_R)\vec{v}_o(t_R)| \tag{19}$$

Call $\vec{\mathcal{R}} = \vec{x}-\vec{x}_o(t_R)$; then

$$|\vec{x}-\vec{x}_o(t')| \approx (\mathcal{R}^2 - 2\vec{\mathcal{R}}\cdot\vec{v}_o(t'-t_R))^{1/2} \tag{20}$$

$$\approx \mathcal{R} - \frac{\vec{\mathcal{R}}\cdot\vec{v}_o}{\mathcal{R}}(t'-t_R)$$

So finally, the argument of the δ-function is:

$$\delta([t-\frac{1}{c}\mathcal{R} - t_R\frac{1}{c}\vec{v}_o\cdot\frac{\vec{\mathcal{R}}}{\mathcal{R}}] - t'(1-\frac{1}{c}\vec{v}_o\cdot\frac{\vec{\mathcal{R}}}{\mathcal{R}})) \tag{21}$$

Remember that t' is the integration variable and note that t' appears only in the second set of terms. The δ function still vanishes when $t' = t_R$. But what we also need is that:

$$\delta(a(t'-t_R)) = \frac{1}{a}\delta(t'-t_R)). \tag{22}$$

So from this we obtain:

$$V(\vec{r},t) = \frac{q}{4\pi\epsilon_o}\frac{1}{\mathcal{R}-\frac{1}{c}\vec{v}_o\cdot\vec{\mathcal{R}}} \tag{23}$$

$$\vec{A}(\vec{r},t) = \mu_o\frac{q\vec{v}}{4\pi}\frac{1}{\mathcal{R}-\frac{1}{c}\vec{v}_o\cdot\vec{\mathcal{R}}} \tag{24}$$

where in each case, the quantities on the right hand side are evaluated at the retarded time.

3 Evaluating the Fields

Our index notation is particularly effective in evaluating the $\vec{E}$ and $\vec{B}$ fields of a point charge. We need to evaluate:

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} V \qquad \vec{B} = \vec{\nabla} \times \vec{A}. \tag{25}$$

We need to be careful, however, because t_R is implicitly a function of $\vec{x}$. So when we take derivatives with respect to $\vec{x}$, we need to differentiate not only the terms with explicit $\vec{x}$'s, but also the terms with t_R. So we start by working out these derivatives. Differentiating both sides of:

$$t_R = t - \frac{1}{c}|\vec{x} - \vec{x}_o(t_R)| \tag{26}$$

remembering that

$$|\vec{x} - \vec{x}_o(t_R)| = ((x_i - x_{oi})^2)^{1/2} \tag{27}$$

gives

$$\partial_i t_R = -\frac{1}{c}\frac{\mathcal{R}_i}{\mathcal{R}} + \frac{\vec{v}_o(t_R)}{c} \cdot \frac{\vec{\mathcal{R}}}{\mathcal{R}} \partial_i t_R \tag{28}$$

Solving for $\partial_i t_R$:

$$\partial_i t_R = -\frac{\mathcal{R}_i}{c\mathcal{R}} \frac{1}{1 - \vec{v}_o(t_R) \cdot \frac{\vec{\mathcal{R}}}{\mathcal{R}}} \tag{29}$$

It will also be useful to have a formula for $\partial_i \mathcal{R}$. From

$$\mathcal{R} = c(t - t_R) \tag{30}$$

we have

$$\partial_i \mathcal{R} = -c\partial_i t_R. \tag{31}$$

So now we can start taking derivatives.

$$\partial_i V = -\frac{c}{4\pi\epsilon(\mathcal{R}c - \vec{\mathcal{R}} \cdot \vec{v})^2} \partial_i(\mathcal{R}c - \vec{\mathcal{R}} \cdot \vec{v}) \tag{32}$$

Now

$$\begin{aligned} \partial_i \vec{\mathcal{R}} \cdot \vec{v} &= \partial_i (r_j - x_{oj}(t_R))\dot{x}_{oj}(t_R) \\ &= \dot{x}_{oi} - \dot{x}_{oj}^2 \partial_i t_R - \mathcal{R}_j \ddot{x}_{oj} \partial_i t_R \end{aligned} \tag{33}$$

So

$$\partial_i V = -\frac{qc}{4\pi\epsilon_o(\mathcal{R}c - \vec{\mathcal{R}} \cdot \vec{v})^2}(-c\partial_i t_R + v^2 \partial_i t_R + \vec{\mathcal{R}} \cdot \vec{a} \partial_i t_R - v_i) \tag{34}$$

Using our expression for $\partial_i t_R$ gives:

$$\partial_i V = \frac{-qc}{4\pi\epsilon_o(\mathcal{R}c - \vec{\mathcal{R}} \cdot \vec{v})^3}\left[-c^2\mathcal{R}_i + v^2\mathcal{R}_i + \vec{\mathcal{R}} \cdot \vec{a}\mathcal{R}_i - v_i(\vec{\mathcal{R}} \cdot \vec{v} - c\mathcal{R})\right] \tag{35}$$

With a bit more algebra, one can show:

$$\frac{\partial \vec{A}}{\partial t} = \frac{1}{4\pi\epsilon_o}\frac{qc}{(\mathcal{R}c - \vec{\mathcal{R}} \cdot \vec{v})^3}\left[(\mathcal{R}c - \vec{\mathcal{R}} \cdot \vec{v})(-\vec{v} + \mathcal{R}\vec{a}/c) + \frac{\mathcal{R}}{c}(c^2 - v^2 + \vec{\mathcal{R}} \cdot \vec{a})\vec{v}\right] \tag{36}$$

and combining these, you obtain:

$$\vec{E}(\vec{r}, t) = \frac{1}{4\pi\epsilon_o}\frac{\mathcal{R}}{(\vec{\mathcal{R}} \cdot \vec{u})^3}\left[(c^2 - v^2)\vec{u} + \vec{\mathcal{R}} \times (\vec{u} \times \vec{a})\right] \tag{37}$$

where $\vec{u} = c\hat{\mathcal{R}} - \vec{v}$. Similarly,

$$\vec{B} = \frac{1}{c}\hat{\mathcal{R}} \times \vec{E}(\vec{r}, t). \tag{38}$$

Exercise: Fill in the details of the calculations of $\vec{E}$ and $\vec{B}$, using the index notation as above.

Where does the energy go?

$$\vec{S} \cdot \hat{r} = \frac{1}{\mu_0}(\vec{E} \times \vec{B}) \cdot \hat{n} \tag{39}$$

$$= \frac{q^2}{4\pi c}\frac{1}{\mathcal{R}^2}\left|\frac{\hat{r} \times (\hat{r} - \vec{\beta}) \times \frac{d\vec{\beta}}{dt}}{(1 - \vec{\beta} \cdot \hat{n})^3}\right|^2$$

Note different behaviors if velocity parallel, perpendicular to acceleration (circular vs. linear motion). Also peaking with angle.