Physics 110B. Electricity and Magnetism. Professor Dine

Spring 2008: MIDTERM EXAM

Exercise 1. A spherical ball of radius R contains charge, and charge is flowing in and out. At the surface, the flux is

$$\vec{J} = \hat{r}\frac{A}{R^2} + \hat{\theta}\frac{B}{R^3}.$$

What is the time rate of change of the charge inside the sphere?

Solution: The total change of charge in the sphere per unit time is obtained by dotting the current with the outward pointing normal, $\hat{r}$, and integrating over the sphere. The result is simply:

$$\dot{Q}_V = -4\pi A$$

Exercise 2. Derive the identity

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla} \times (\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A}$$

using the index notation.

Solution: This we've done many times.

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A})_i = \epsilon_{ijk}\partial_j \epsilon_{klm}\partial_l A_m$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j\partial_l A_m$$

$$= \partial_i\partial_j A_j - \partial_j\partial_j A_i$$

which is the component form of the identity.

Problem 1. A plane wave moves in vacuum along the z axis, with polarization in the x direction, with $\vec{E}$ field $\vec{E} = E_0\hat{x}e^{ikz-i\omega t}$.

a. Determine the magnetic field.

Solution:

Use Faraday's law, $\vec{\nabla} \times \vec{B} = -\frac{\partial \vec{B}}{\partial t}$ to give

$$i\vec{k} \times \vec{E}_0 = i\omega \vec{B}_0$$

so

$$\vec{B} = \frac{\vec{k} \times \vec{E}_0}{\omega} e^{ikz-i\omega t}$$

where $\vec{k} = k\hat{z}$, so

$$\vec{B} = \frac{1}{c}E_0\hat{y}e^{ikz-i\omega t}$$

b. Determine the energy density and energy flux.

Solution:

Here we just plug in, remembering to time average. The result is

$$u = \frac{1}{2}\epsilon_0 \vec{E}_0^2$$

$$\vec{S} = \frac{1}{2}\frac{1}{c\epsilon_0\mu_0}u = cu$$

Problem 2. In class, we have just encountered the formula:

$$\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}(= -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}).$$

(the alternative notation which I like is in parenthesis). Consider an alternative description of the plane wave of problem 1.

$$\vec{A} = \vec{A}_0 e^{i\vec{k}\cdot\vec{x} - i\omega t} \qquad V = 0(\phi = 0).$$

a. Suppose that we work in the *gauge* $\vec{\nabla} \cdot \vec{A} = 0$. What does this mean for $\vec{A}_0$? Calculate the electric field.

Solution:

$$\vec{k} \cdot \vec{A}_0 = 0,$$

i.e. $\vec{A}_0$ is orthogonal to $\vec{k}$. As we see below, this means that $\vec{E}$, which is along $\vec{A}_0$, is perpendicular to $\vec{k}$ (transverse), and $\vec{B}$ is transverse as well.

$$\vec{E} = i\omega \vec{A}_0 e^{i\vec{k}\cdot\vec{x} - i\omega t}$$

$$\begin{aligned}\vec{B} &= i\vec{k} \times \vec{A}_0 e^{i\vec{k}\cdot\vec{x} - i\omega t} \\ &= \frac{1}{c}\hat{k} \times \vec{E}\end{aligned}$$

b. Calculate the magnetic field from $\vec{B} = \vec{\nabla} \times \vec{A}$.

Solution:

c. For what values of $\vec{k}$ and $\vec{A}_0$ does this set of potentials describe the wave of problem 1?

Solution:

$$\vec{k} = k\hat{z}; \vec{A}_0 = -i\hat{x}E_0/\omega.$$

Problem 3. A wave passes through a dielectric material with dielectric constant ϵ (independent of frequency) and $\mu = \mu_0$, and without free charges or currents.

a. From Maxwell's equations for the medium, derive the wave equation appropriate for this situation. (You may use the vector identities below, even if you did not succeed with exercise 2). What is the speed of light in the medium?

Solution: Take curl of the equation for $\vec{\nabla}\vec{H}$, and use $\vec{\nabla} \cdot \vec{H} = 0$, and the equation for $\vec{\nabla} \times \vec{D}$, to give

$$-\vec{\nabla}^2 \vec{B} = -\epsilon \frac{\partial^2 \vec{B}}{\partial t^2}$$

so

$$v^2/c^2 = \frac{\epsilon_0}{\epsilon}$$

b. Suppose now that the dielectric constant is a function of frequency, $\epsilon/\epsilon_0 = 1 + \frac{Nq^2}{m\epsilon_0}\frac{1}{(\omega_0^2-\omega^2)-i\omega\gamma}$ where $\gamma \ll \omega_0$. Discuss the propagation of the waves for very low frequency and very high frequency, neglecting gamma (i.e. describe the index of refraction, velocity of propagation.

Solution: In the very high frequency limit, $\epsilon = \epsilon_0$ and $n = 1$, i.e. waves propagate at the speed of light (in vacuum) with no absorption. At very low frequencies, $\epsilon/\epsilon_0 = 1 + C$, where

$$C = \frac{Nq^2}{m\epsilon_0}\frac{1}{\omega_0^2}$$

i.e. there is a correction, which is real (no absorption) and independent of frequency for low enough frequency.

c. Now include the small constant, γ, and discuss the attenuation of the waves in the medium at low frequency. You can assume that the real corrections to ϵ are small.

Solution: Now we have

$$\epsilon/\epsilon_0 = 1 + \frac{Nq^2}{m\epsilon_0}\frac{1}{(\omega_0^2-\omega^2)(1-\frac{i\gamma\omega}{(\omega_0^2-\omega^2)})}$$

$$\approx 1 + \frac{Nq^2}{m\epsilon_0}\left(\frac{1}{(\omega_0^2-\omega^2)} + i\frac{\gamma\omega}{(\omega_0^2-\omega^2)^2}\right)$$

Taking the square root puts a factor of 1/2 in front. Writing $k = \frac{1}{v}\omega = n\omega$, the ik becomes negative, corresponding to absorption.

d. Consider now frequencies very near ω_0 ("resonance"). Construct the real part of the ϵ/ϵ_0. Discuss the phase velocity just above and below the resonance. Is it ever greater than c? Do waves ever propagate faster than the speed of light? Explain (words are enough; you do not have to do a calculation).

Solution: Now real part of ϵ is

$$\epsilon = 1 + \frac{Nq^2}{m\epsilon_0}\frac{\omega_0^2-\omega^2}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}.$$

The correction is negative above the resonant frequency, so the phase velocity becomes greater than the speed of light. But as you saw in your homework, the group velocity does not.

Some Useful Formulas

1. Snell's law:

$$\frac{\sin(\theta_T)}{\sin(\theta_I)} = \frac{n_1}{n_2}$$

2.

$$n = \sqrt{\epsilon\mu} = \frac{1}{v}c$$

3. Maxwell's Equations in a medium ($D = \epsilon E$,$\vec{B} = \mu\vec{H}$):

$$\vec{\nabla}\cdot\vec{D} = \rho_f \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial\vec{D}}{\partial t}$$

where ρ_f and $\vec{J}_f$ are the free charge and current densities.

4.

$$(\vec{A}\times\vec{B})_i = \epsilon_{ijk}A_jB_k$$

5.

$$\epsilon_{ijk}\epsilon_{k\ell m} = (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})$$

6.

$$\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t} \qquad \vec{B} = \vec{\nabla}\times\vec{A}$$

7. Energy density of the electromagnetic field:

$$u = \frac{1}{2}\left(\epsilon_0\vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2\right)$$

8. Poynting vector:

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$

9. Momentum density

$$\vec{P} = \mu_0\epsilon_0\vec{S}$$