Physics 110B. Electricity and Magnetism. Professor Dine

Spring 2008: QUIZ 2

Problem 1. Practice with vectors..
a. Calculate the gradient of $r = |\vec{x}|$, using the formula for the gradient in spherical coordinates below.
Solution: In eqn. 5 below, only the first term is non-vanishing; $\vec{\nabla} r = \hat{r}$.
b. Do the same using Cartesian coordinates and the index notation.
Solution:

$$\partial_i \sqrt{x_j x_j} = \frac{x_i}{\sqrt{x_j x_j}} = \hat{r}_i \tag{1}$$

giving same result as above.
Problem 2. Spherical waves.
a. Explain why $f(r) = e^{ikr - i\omega t}$ solves the wave equation if r is large. Use the formula for the Laplacian in spherical coordinates (see below).
Solution: In eqn. 6, only need do keep first term, and only terms where the derivatives act on the exponential. So

$$\nabla^2 f = -k^2 f \tag{2}$$

So the wave equation is solved if $\omega = ck$.
b. Do the same using Cartesian coordinates and the index notation.
Problem 3. We saw that the electric field of a point charge is:

$$\vec{E}(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\mathcal{R}}{(\vec{\mathcal{R}} \cdot \vec{u})^3} \left[(c^2 - v^2)\vec{u} + \vec{\mathcal{R}} \times (\vec{u} \times \vec{a}) \right]. \tag{3}$$

where $\vec{u} = c\hat{\mathcal{R}} - \vec{v}$, and $\mathcal{R} = \vec{r} - \vec{x}_0(t_r)$.
a. Which of the two terms in brackets describes radiation? Why?
Solution: Second term since it falls as $1/r$ for large r. Note it is the only term involving the acceleration.
b. In the limit of zero velocity, show this reduces to the formula you know so well.
Solution: Setting $v = 0$, we have $\vec{u} = c\hat{R}$, and so we get

$$\vec{E} = \frac{1}{4\pi\epsilon_0 \mathcal{R}^3} \vec{R}. \tag{4}$$

This is the familiar Coulomb form.

Formulas You Might Need

1.
$$(\vec{A} \times \vec{B})_i = \epsilon_{ijk} A_j B_k \tag{5}$$

2.
$$\epsilon_{ijk}\epsilon_{k\ell m} = (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}) \tag{6}$$

3.
$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \vec{\nabla} \times \vec{A} \tag{7}$$

4. Gradient in spherical coordinates:

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}. \tag{8}$$

5. Laplacian in spherical coordinates:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial f}{\partial \phi^2}. \tag{9}$$