Problem 1. Practice with vectors.
a. Calculate the gradient of \( r = |\vec{x}| \), using the formula for the gradient in spherical coordinates below.
Solution: In eqn. 5 below, only the first term is non-vanishing; \( \nabla r = \hat{r} \).
b. Do the same using Cartesian coordinates and the index notation.
Solution:
\[
\frac{\partial_i \sqrt{x_j x_j}}{\sqrt{x_j x_j}} = \hat{r}_i
\]  
(1)
giving same result as above.

Problem 2. Spherical waves.
a. Explain why \( f(r) = e^{ikr - i\omega t} \) solves the wave equation if \( r \) is large. Use the formula for the Laplacian in spherical coordinates (see below).
Solution: In eqn. 6, only need do keep first term, and only terms where the derivatives act on the exponential. So
\[
\nabla^2 f = -k^2 f
\]  
(2)
So the wave equation is solved if \( \omega = ck \).
b. Do the same using Cartesian coordinates and the index notation.

Problem 3. We saw that the electric field of a point charge is:
\[
\vec{E}(\vec{r}, t) = \frac{q}{4\pi \epsilon_0} \frac{\vec{R}}{(\vec{R} \cdot \vec{u})^3} \left[ (c^2 - v^2)\vec{u} + \vec{R} \times (\vec{u} \times \vec{a}) \right].
\]  
(3)
where \( \vec{u} = c\hat{R} - \vec{v} \), and \( \vec{R} = \vec{r} - \vec{x}_0(t_r) \).
a. Which of the two terms in brackets describes radiation? Why?
Solution: Second term since it falls as \( 1/r \) for large \( r \). Note it is the only term involving the acceleration.
b. In the limit of zero velocity, show this reduces to the formula you know so well.
Solution: Setting \( v = 0 \), we have \( \vec{u} = c\hat{R} \), and so we get
\[
\vec{E} = \frac{1}{4\pi \epsilon_0 \vec{R}^3} \vec{R}.
\]  
(4)
This is the familiar Coulomb form.

Formulas You Might Need

1.
\[
(\vec{A} \times \vec{B})_i = \epsilon_{ijk} A_j B_k
\]  
(5)
2.
\[
\epsilon_{ijk} \epsilon_{k\ell m} = (\delta_{i\ell} \delta_{jm} - \delta_{im} \delta_{j\ell})
\]  
(6)
3.
\[
\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \quad \vec{B} = \nabla \times \vec{A}
\]  
(7)
4. Gradient in spherical coordinates:

\[ \vec{\nabla} f = \frac{\partial f}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\phi}. \]  

(8)

5. Laplacian in spherical coordinates:

\[ \nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}. \]  

(9)