Physics 110. Electricity and Magnetism. Professor Dine

Spring, 2008. Handout: Poynting Vector and Stress Tensor

Poynting Vector

We derived the energy density and the energy flux of the electromagnetic field:

$$u = \frac{1}{2}\left(\epsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2\right) \tag{1}$$

and

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}. \tag{2}$$

We worked this out for our plane wave solution:

$$E_x = A\cos(kz - \omega t) \qquad B_y = \frac{k}{\omega}A\cos(kz - \omega t) \tag{3}$$

where $\omega = k/\sqrt{\epsilon_0 \mu_0}$. Then

$$u = A^2\cos^2(kz - \omega t)\frac{1}{2}\left(\epsilon_0 + \frac{1}{\mu_0}\epsilon_0\mu_0.\right) \tag{4}$$

$$= \epsilon_0 A^2 \cos^2(kz - \omega t)$$

whereas

$$\vec{S} = \frac{1}{\mu^0}A^2\cos^2(kz - \omega t)\sqrt{\epsilon_0\mu_0}\hat{z} \tag{5}$$

so

$$\vec{S} = \frac{1}{\sqrt{\epsilon_0\mu_0}}u\hat{z} = cu\hat{z}. \tag{6}$$

So the flux of energy, is just the energy density times the velocity at which the wave moves.

The Maxwell Stress Tensor – some practice with our index methods

From the Lorentz force law and Maxwell's equations, we derived the expression:

$$\vec{f} = \frac{d\vec{p}}{dt} = \int d\tau \left(-\epsilon_0 \frac{d}{dt}(\vec{E} \times \vec{B}) + \epsilon_0 \left[(\vec{\nabla} \cdot \vec{E})\vec{E} - \vec{E} \times (\vec{\nabla} \times \vec{E}) \right] - \frac{1}{\mu_0} \left[(\vec{\nabla} \cdot \vec{B})\vec{B} - \vec{B} \times (\vec{\nabla} \times \vec{B}) \right] \right). \tag{7}$$

Now we want to write this so it looks like another conservation equation: one derivative term, one divergence term. For this, our index notation is useful. Start with:

$$f_i = \int d\tau \, (-\epsilon_0 \mu_0 S_i + h_i) \tag{8}$$

with

$$h_i = \epsilon_0 \left[\partial_j E_j E_i - \epsilon_{ijk} E_j \epsilon_{klm} \partial_\ell E_m \right] + \frac{1}{\mu_0} \left[\partial_j B_j B_i - \epsilon_{ijk} B_j \epsilon_{klm} \partial_\ell B_m \right]. \tag{9}$$

Using our familiar identity,

$$\begin{aligned} \epsilon_{ijk} E_j \epsilon_{klm} \partial_\ell E_m &= (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{jl}) E_j \partial_\ell E_m \qquad (10) \\ &= E_j \partial_i E_j - E_j \partial_j E_i \\ &= \frac{1}{2} \partial_i (\vec{E}^2) - E_j \partial_j E_i \end{aligned}$$

The term with two ϵ's and $\vec{B}$ is similar, so we have

$$h_i = -\frac{1}{2} \partial_i (\epsilon_0 E_j^2 + \frac{1}{\mu_0} B_j^2) + \epsilon_0 \left[(\partial_j E_j) E_i + E_j \partial_j E_i \right] + \frac{1}{\mu_0} \epsilon_0 \left[(\partial_j B_j) B_i + B_j \partial_j B_i \right] \tag{11}$$

Now this looks almost like what we want; the second term is a divergence (of something with an i index!). But we can write the first term in the form of a divergence by judicious use of Kronecker delta's. For example,

$$\partial_i E_j^2 = \partial_j \delta_{ij} E_k^2 \tag{12}$$

In this way, if we define:

$$T_{ij} = \epsilon_0 (E_i E_j - \frac{1}{2} \delta_{ij} \vec{E}^2) + \frac{1}{\mu_0} (B_i B_j - \frac{1}{2} \delta_{ij} \vec{B}^2) \tag{13}$$

then

$$h_i = \partial_j T_{ij} \tag{14}$$

and we have

$$\left(\frac{dP_{EM}}{dt} \right)_i = \partial_j T_{ij}. \tag{15}$$

Here

$$\vec{P}_{EM} = \epsilon_0 \mu_0 \vec{S}. \tag{16}$$