The Feynman LECTURES ON PHYSICS

MAINLY ELECTROMAGNETISM AND MATTER

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The advantages are much less clear for magnetostatics. The integral for $\boldsymbol{A}$ is already a vector integral:

$$\boldsymbol{A}(1) = \frac{1}{4\pi\epsilon_0 c^2}\int \frac{\boldsymbol{j}(2)\,dV_2}{r_{12}}, \tag{15.24}$$

which is, of course, three integrals. Also, when we take the curl of $\boldsymbol{A}$ to get $\boldsymbol{B}$, we have six derivatives to do and combine by pairs. It is not immediately obvious whether in most problems this procedure is really any easier than computing $\boldsymbol{B}$ directly from

$$\boldsymbol{B}(1) = \frac{1}{4\pi\epsilon_0 c^2}\int \frac{\boldsymbol{j}(2)\times\boldsymbol{e}_{12}}{r_{12}^2}\,dV_2. \tag{15.25}$$

Using the vector potential is often more difficult for simple problems for the following reason. Suppose we are interested only in the magnetic field $\boldsymbol{B}$ at one point, and that the problem has some nice symmetry—say we want the field at a point on the axis of a ring of current. Because of the symmetry, we can easily get $\boldsymbol{B}$ by doing the integral of Eq. (15.25). If, however, we were to find $\boldsymbol{A}$ first, we would have to compute $\boldsymbol{B}$ from *derivatives* of $\boldsymbol{A}$, so we must know what $\boldsymbol{A}$ is at all points in the *neighborhood* of the point of interest. And most of these points are off the axis of symmetry, so the integral for $\boldsymbol{A}$ gets complicated. In the ring problem, for example, we would need to use elliptic integrals. In such problems, $\boldsymbol{A}$ is clearly not very useful. It is true that in many complex problems it is easier to work with $\boldsymbol{A}$, but it would be hard to argue that this ease of technique would justify making you learn about one more vector field.

We have introduced $\boldsymbol{A}$ because it *does* have an important physical significance. Not only is it related to the energies of currents, as we saw in the last section, but it is also a "real" physical field in the sense that we described above. In classical mechanics it is clear that we can write the force on a particle as

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}), \tag{15.26}$$

so that, given the forces, everything about the motion is determined. In any region where $\boldsymbol{B} = 0$ even if $\boldsymbol{A}$ is not zero, such as outside a solenoid, there is no discernible effect of $\boldsymbol{A}$. Therefore for a long time it was believed that $\boldsymbol{A}$ was not a "real" field. It turns out, however, that there are phenomena involving quantum mechanics which show that the field $\boldsymbol{A}$ is in fact a "real" field in the sense we have defined it. In the next section we will show you how that works.

15–5 The vector potential and quantum mechanics

There are many changes in what concepts are important when we go from classical to quantum mechanics. We have already discussed some of them in Vol. I. In particular, the force concept gradually fades away, while the concepts of energy and momentum become of paramount importance. You remember that instead of particle motions, one deals with probability amplitudes which vary in space and time. In these amplitudes there are wavelengths related to momenta, and frequencies related to energies. The momenta and energies, which determine the phases of wave functions, are therefore the important quantities in quantum mechanics. Instead of forces, we deal with the way interactions change the wavelength of the waves. The idea of a force becomes quite secondary—if it is there at all. When people talk about nuclear forces, for example, what they usually analyze and work with are the energies of interaction of two nucleons, and not the force between them. Nobody ever differentiates the energy to find out what the force looks like. In this section we want to describe how the vector and scalar potentials enter into quantum mechanics. It is, in fact, just because momentum and energy play a central role in quantum mechanics that $\boldsymbol{A}$ and ϕ provide the most direct way of introducing electromagnetic effects into quantum descriptions.

We must review a little how quantum mechanics works. We will consider again the imaginary experiment described in Chapter 37 of Vol. I, in which elec-

2

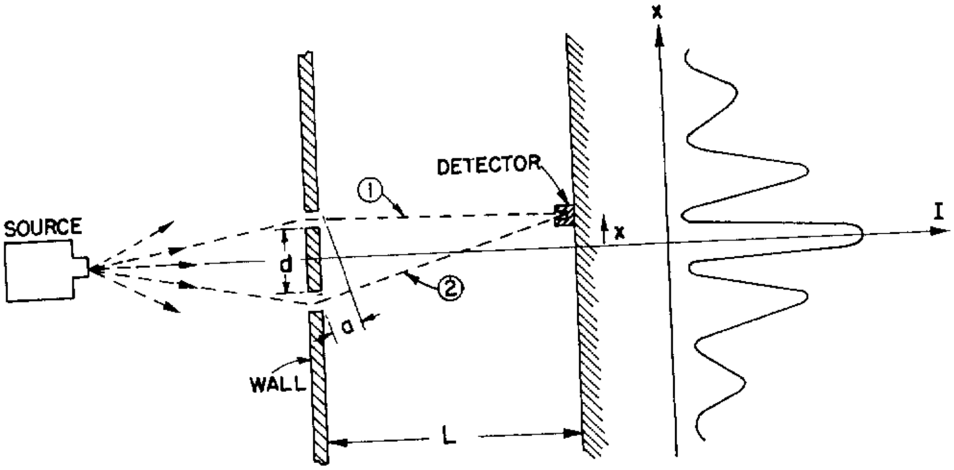

Fig. 15–5. An interference experiment with electrons (see also Chapter 37 of Vol. I).

trons are diffracted by two slits. The arrangement is shown again in Fig. 15–5. Electrons, all of nearly the same energy, leave the source and travel toward a wall with two narrow slits. Beyond the wall is a "backstop" with a movable detector. The detector measures the rate, which we call I, at which electrons arrive at a small region of the backstop at the distance x from the axis of symmetry. The rate is proportional to the probability that an individual electron that leaves the source will reach that region of the backstop. This probability has the complicated-looking distribution shown in the figure, which we understand as due to the interference of two amplitudes, one from each slit. The interference of the two amplitudes depends on their phase difference. That is, if the amplitudes are $C_1e^{i\Phi_1}$ and $C_2e^{i\Phi_2}$, the phase difference $\delta = \Phi_1 - \Phi_2$ determines their interference pattern [see Eq. (29.12) in Vol. I]. If the distance between the screen and the slits is L, and if the difference in the path lengths for electrons going through the two slits is a, as shown in the figure, then the phase difference of the two waves is given by

$$\delta = \frac{a}{\lambda\!\!\!^{-}}. \tag{15.27}$$

As usual, we let $\lambda\!\!\!^{-} = \lambda/2\pi$, where λ is the wavelength of the space variation of the probability amplitude. For simplicity, we will consider only values of x much less than L; then we can set

$$a = \frac{x}{L}\,d$$

and

$$\delta = \frac{x}{L}\frac{d}{\lambda\!\!\!^{-}}. \tag{15.28}$$

When x is zero, δ is zero; the waves are in phase, and the probability has a maximum. When δ is π, the waves are out of phase, they interfere destructively, and the probability is a minimum. So we get the wavy function for the electron intensity.

Now we would like to state the law that for quantum mechanics replaces the force law $\boldsymbol{F} = q\boldsymbol{v}\times\boldsymbol{B}$. It will be the law that determines the behavior of quantum-mechanical particles in an electromagnetic field. Since what happens is determined by amplitudes, the law must tell us how the magnetic influences affect the amplitudes; we are no longer dealing with the acceleration of a particle. The law is the following: the phase of the amplitude to arrive via any trajectory is changed by the presence of a magnetic field by an amount equal to the integral of the vector potential along the whole trajectory times the charge of the particle over Planck's constant. That is,

$$\text{Magnetic change in phase} = \frac{q}{\hbar}\int_{\text{trajectory}} \boldsymbol{A}\cdot d\boldsymbol{s}. \tag{15.29}$$

3

If there were no magnetic field there would be a certain phase of arrival. If there is a magnetic field anywhere, the phase of the arriving wave is increased by the integral in Eq. (15.29).

Although we will not need to use it for our present discussion, we mention that the effect of an electrostatic field is to produce a phase change given by the *negative* of the *time* integral of the scalar potential ϕ:

$$\text{Electric change in phase} = -\frac{q}{\hbar}\int \phi\, dt.$$

These two expressions are correct not only for static fields, but together give the correct result for *any* electromagnetic field, static or dynamic. This is the law that replaces $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$. We want now, however, to consider only a static magnetic field.

Suppose that there is a magnetic field present in the two-slit experiment. We want to ask for the phase of arrival at the screen of the two waves whose paths pass through the two slits. Their interference determines where the maxima in the probability will be. We may call Φ_1 the phase of the wave along trajectory (1). If $\Phi_1(B = 0)$ is the phase without the magnetic field, then when the field is turned on the phase will be

$$\Phi_1 = \Phi_1(B = 0) + \frac{q}{\hbar}\int_{(1)} \boldsymbol{A} \cdot d\boldsymbol{s}. \tag{15.30}$$

Similarly, the phase for trajectory (2) is

$$\Phi_2 = \Phi_2(B = 0) + \frac{q}{\hbar}\int_{(2)} \boldsymbol{A} \cdot d\boldsymbol{s}. \tag{15.31}$$

The interference of the waves at the detector depends on the phase difference

$$\delta = \Phi_1(B = 0) - \Phi_2(B = 0) + \frac{q}{\hbar}\int_{(1)} \boldsymbol{A} \cdot d\boldsymbol{s} - \frac{q}{\hbar}\int_{(2)} \boldsymbol{A} \cdot d\boldsymbol{s}. \tag{15.32}$$

The no-field difference we will call $\delta(B = 0)$; it is just the phase difference we have calculated above in Eq. (15.28). Also, we notice that the two integrals can be written as *one* integral that goes forward along (1) and back along (2); we call this the closed path (1–2). So we have

$$\delta = \delta(B = 0) + \frac{q}{\hbar}\oint_{(1-2)} \boldsymbol{A} \cdot d\boldsymbol{s}. \tag{15.33}$$

This equation tells us how the electron motion is changed by the magnetic field; with it we can find the new positions of the intensity maxima and minima at the backstop.

Before we do that, however, we want to raise the following interesting and important point. You remember that the vector potential function has some arbitrariness. Two different vector potential functions $\boldsymbol{A}$ and $\boldsymbol{A}'$ whose difference is the gradient of some scalar function $\nabla\psi$, both represent the same magnetic field, since the curl of a gradient is zero. They give, therefore, the same classical force $q\boldsymbol{v} \times \boldsymbol{B}$. If in quantum mechanics the effects depend on the vector potential, *which* of the many possible $\boldsymbol{A}$-functions is correct?

The answer is that the same arbitrariness in $\boldsymbol{A}$ continues to exist for quantum mechanics. If in Eq. (15.33) we change $\boldsymbol{A}$ to $\boldsymbol{A}' = \boldsymbol{A} + \nabla\psi$, the integral on $\boldsymbol{A}$ becomes

$$\oint_{(1-2)} \boldsymbol{A}' \cdot d\boldsymbol{s} = \oint_{(1-2)} \boldsymbol{A} \cdot d\boldsymbol{s} + \oint_{(1-2)} \nabla\psi \cdot d\boldsymbol{s}.$$

The integral of $\nabla\psi$ is around the *closed* path (1–2), but the integral of the tangential component of a gradient on a closed path is always zero, by Stokes' theorem. Therefore both $\boldsymbol{A}$ and $\boldsymbol{A}'$ give the same phase differences and the same quantum-mechanical interference effects. In both classical and quantum theory it is only the curl of $\boldsymbol{A}$ that matters; any choice of the function of $\boldsymbol{A}$ which has the correct curl gives the correct physics.

The same conclusion is evident if we use the results of Section 14-1. There we found that the line integral of $\boldsymbol{A}$ around a closed path is the flux of $\boldsymbol{B}$ through the path, which here is the flux between paths (1) and (2). Equation (15.33) can, if we wish, be written as

$$\delta = \delta(B = 0) + \frac{q}{\hbar}\,[\text{flux of } \boldsymbol{B} \text{ between (1) and (2)}], \tag{15.34}$$

where by the flux of $\boldsymbol{B}$ we mean, as usual, the surface integral of the normal component of $\boldsymbol{B}$. The result depends only on $\boldsymbol{B}$, and therefore only on the curl of $\boldsymbol{A}$.

Now because we can write the result in terms of $\boldsymbol{B}$ as well as in terms of $\boldsymbol{A}$, you might be inclined to think that the $\boldsymbol{B}$ holds its own as a "real" field and that the $\boldsymbol{A}$ can still be thought of as an artificial construction. But the definition of "real" field that we originally proposed was based on the idea that a "real" field would not act on a particle from a distance. We can, however, give an example in which $\boldsymbol{B}$ is zero—or at least arbitrarily small—at any place where there is some chance to find the particles, so that it is not possible to think of it acting *directly* on them.

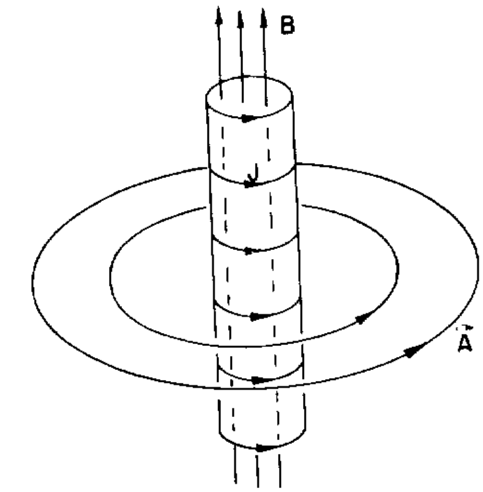

Fig. 15-6. The magnetic field and vector potential of a long solenoid.

You remember that for a long solenoid carrying an electric current there is a $\boldsymbol{B}$-field inside but none outside, while there is lots of $\boldsymbol{A}$ circulating around outside, as shown in Fig. 15-6. If we arrange a situation in which electrons are to be found only *outside* of the solenoid—only where there is $\boldsymbol{A}$—there will still be an influence on the motion, according to Eq. (15.33). Classically, that is impossible. Classically, the force depends only on $\boldsymbol{B}$; in order to know that the solenoid is carrying current, the particle must go through it. But quantum-mechanically you can find out that there is a magnetic field inside the solenoid by going *around* it—without ever going close to it!

Suppose that we put a very long solenoid of small diameter just behind the wall and between the two slits, as shown in Fig. 15-7. The diameter of the solenoid is to be much smaller than the distance d between the two slits. In these circumstances, the diffraction of the electrons at the slit gives no appreciable probability that the electrons will get near the solenoid. What will be the effect on our interference experiment?

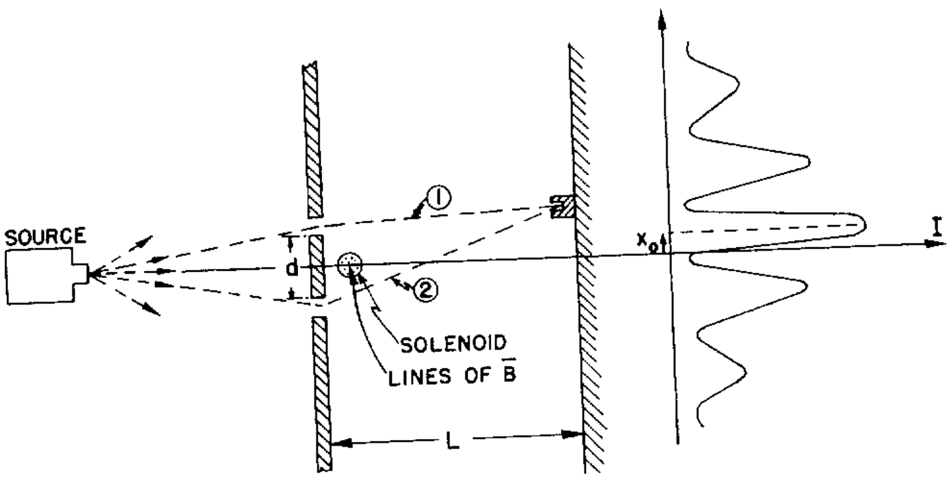

Fig. 15-7. A magnetic field can influence the motion of electrons even though it exists only in regions where there is an arbitrarily small probability of finding the electrons.

We compare the situation with and without a current through the solenoid. If we have no current, we have no $\boldsymbol{B}$ or $\boldsymbol{A}$ and we get the original pattern of electron intensity at the backstop. If we turn the current on in the solenoid and build up a magnetic field $\boldsymbol{B}$ inside, then there is an $\boldsymbol{A}$ outside. There is a shift in the phase difference proportional to the circulation of $\boldsymbol{A}$ outside the solenoid, which will mean that the pattern of maxima and minima is shifted to a new position. In fact, since the flux of $\boldsymbol{B}$ inside is a constant for any pair of paths, so also is the circulation of $\boldsymbol{A}$. For every arrival point there is the same phase change; this corresponds

5

to shifting the entire pattern in x by a constant amount, say x_0, that we can easily calculate. The maximum intensity will occur where the phase difference between the two waves is zero. Using Eq. (15.32) or Eq. (15.33) for δ and Eq. (15.28) for $\delta(B = 0)$, we have

$$x_0 = -\frac{L}{d}\,\lambdabar\,\frac{q}{\hbar}\oint_{(1-2)} \boldsymbol{A}\cdot d\boldsymbol{s}, \tag{15.35}$$

or

$$x_0 = -\frac{L}{d}\,\lambdabar\,\frac{q}{\hbar}\,[\text{flux of } \boldsymbol{B} \text{ between (1) and (2)}]. \tag{15.36}$$

The pattern with the solenoid in place should appear* as shown in Fig. 15–7. At least, that is the prediction of quantum mechanics.

Precisely this experiment has recently been done. It is a very, very difficult experiment. Because the wavelength of the electrons is so small, the apparatus must be on a tiny scale to observe the interference. The slits must be very close together, and that means that one needs an exceedingly small solenoid. It turns out that in certain circumstances, iron crystals will grow in the form of very long, microscopically thin filaments called whiskers. When these iron whiskers are magnetized they are like a tiny solenoid, and there is no field outside except near the ends. The electron interference experiment was done with such a whisker between two slits, and the predicted displacement in the pattern of electrons was observed.

In our sense then, the $\boldsymbol{A}$-field is "real." You may say: "But there *was* a magnetic field." There was, but remember our original idea—that a field is "real" if it is what must be specified *at the position* of the particle in order to get the motion. The $\boldsymbol{B}$-field in the whisker acts at a distance. If we want to describe its influence not as action-at-a-distance, we must use the vector potential.

This subject has an interesting history. The theory we have described was known from the beginning of quantum mechanics in 1926. The fact that the vector potential appears in the wave equation of quantum mechanics (called the Schrödinger equation) was obvious from the day it was written. That it cannot be replaced by the magnetic field in any easy way was observed by one man after the other who tried to do so. This is also clear from our example of electrons moving in a region where there is no field and being affected nevertheless. But because in classical mechanics $\boldsymbol{A}$ did not appear to have any direct importance and, furthermore, because it could be changed by adding a gradient, people repeatedly said that the vector potential had no direct physical significance—that only the magnetic and electric fields are "right" even in quantum mechanics. It seems strange in retrospect that no one thought of discussing this experiment until 1956, when Bohm and Aharanov first suggested it and made the whole question crystal clear. The implication was there all the time, but no one paid attention to it. Thus many people were rather shocked when the matter was brought up. That's why someone thought it would be worth while to do the experiment to see that it really was right, even though quantum mechanics, which had been believed for so many years, gave an unequivocal answer. It is interesting that something like this can be around for thirty years but, because of certain prejudices of what is and is not significant, continues to be ignored.

Now we wish to continue in our analysis a little further. We will show the connection between the quantum-mechanical formula and the classical formula—to show why it turns out that if we look at things on a large enough scale it will look as though the particles are acted on by a force equal to $q\boldsymbol{v} \times$ the curl of $\boldsymbol{A}$. To get classical mechanics from quantum mechanics, we need to consider cases in which all the wavelengths are very small compared with distances over which external conditions, like fields, vary appreciably. We shall not prove the result in great generality, but only in a very simple example, to show how it works. Again we consider the same slit experiment. But instead of putting all the magnetic field in a very tiny region between the slits, we imagine a magnetic field that extends

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* If the field $\boldsymbol{B}$ comes out of the plane of the figure, the flux as we have defined it is negative and x_0 is positive.

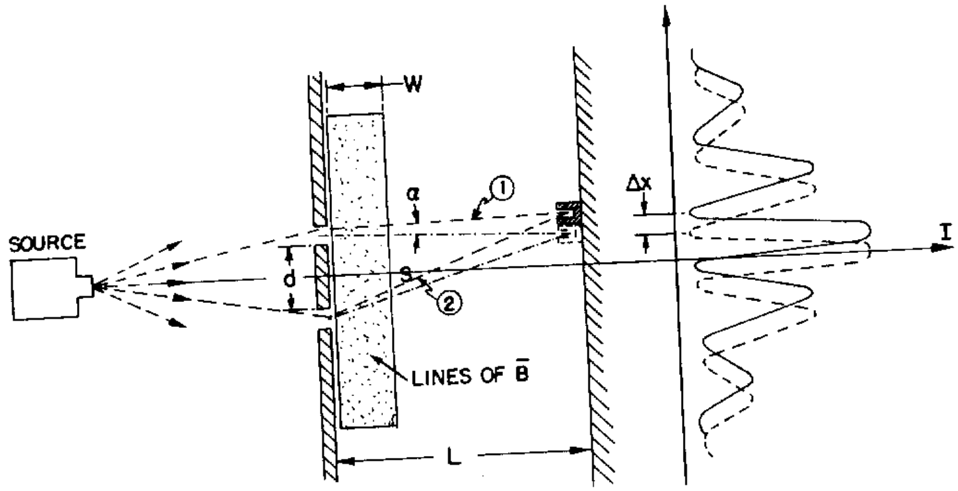

Fig. 15–8. The shift of the interference pattern due to a strip of magnetic field.

over a larger region behind the slits, as shown in Fig. 15–8. We will take the idealized case where we have a magnetic field which is uniform in a narrow strip of width w, considered small as compared with L. (That can easily be arranged; the backstop can be put as far out as we want.) In order to calculate the shift in phase, we must take the two integrals of $\boldsymbol{A}$ along the two trajectories (1) and (2). They differ, as we have seen, merely by the flux of $\boldsymbol{B}$ between the paths. To our approximation, the flux is Bwd. The phase difference for the two paths is then

$$\delta = \delta(B = 0) + \frac{q}{\hbar} Bwd. \tag{15.37}$$

We note that, to our approximation, the phase shift is independent of the angle. So again the effect will be to shift the whole pattern upward by an amount Δx. Using Eq. (15.28),

$$\Delta x = \frac{L\lambda\!\!\!^{-}}{d}\,\Delta\delta = \frac{L\lambda\!\!\!^{-}}{d}\,[\delta - \delta(B = 0)].$$

Using (15.37) for $\delta - \delta(B = 0)$,

$$\Delta x = L\lambda\!\!\!^{-}\,\frac{q}{\hbar}\,Bw. \tag{15.38}$$

Such a shift is equivalent to deflecting all the trajectories by the small angle α (see Fig. 15–8), where

$$\alpha = \frac{\Delta x}{L} = \frac{\lambda\!\!\!^{-}}{\hbar}\,qBw. \tag{15.39}$$

Now classically we would also expect a thin strip of magnetic field to deflect all trajectories through some small angle, say α', as shown in Fig. 15–9(a). As the electrons go through the magnetic field, they feel a transverse force $q\boldsymbol{v} \times \boldsymbol{B}$ which lasts for a time w/v. The change in their transverse momentum is just equal to this impulse, so

$$\Delta p_x = qwB. \tag{15.40}$$

The angular deflection [Fig. 15–9(b)] is equal to the ratio of this transverse momentum to the total momentum p. We get that

$$\alpha' = \frac{\Delta p_x}{p} = \frac{qwB}{p}. \tag{15.41}$$

We can compare this result with Eq. (15.39), which gives the same quantity computed quantum-mechanically. But the connection between classical mechanics and quantum mechanics is this: A particle of momentum p corresponds to a quan-

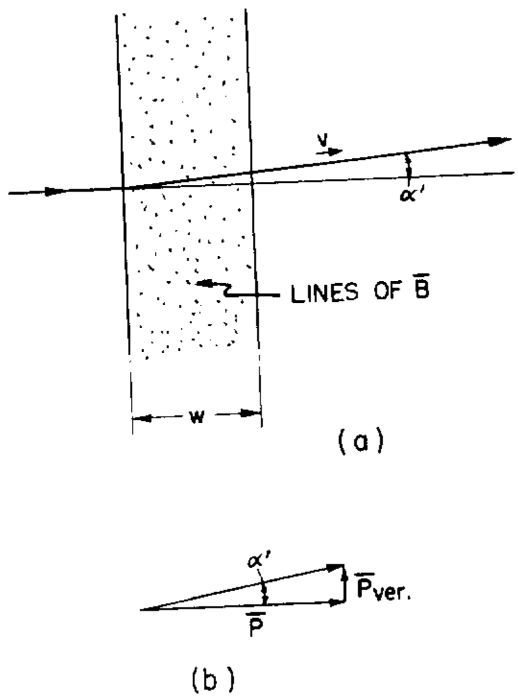

Fig. 15–9. Deflection of a particle due to passage through a strip of magnetic field.