Consider a particle in one dimension, with a lagrangian:

\[ L = \frac{1}{2} m \dot{x}^2 - V(x). \]  

(1)

We have seen that the action principle gives a differential equation for \( x \), which is just Newton’s equation. But it is not hard, in the case of a free particle, to actually compute the action for all possible paths, and verify that the classical solution gives the minimum value of the action.

Suppose we have a particle which starts at time \( t_1 = 0 \) at \( x_1 = 0 \), and at time \( T \) sits at \( x_2 = vT \). Then the classical path is:

\[ x_{cl}(t) = vt. \]  

(2)

Now we want to consider some other path. This can be written as

\[ x(t) = x_{cl}(t) + \delta x(t). \]  

(3)

Because we specify the initial and final position of the particle, and \( x_{cl} \) satisfies these conditions, we have:

\[ \delta x(0) = \delta x(T) = 0. \]  

(4)

So we can expand \( \delta x(t) \) in a Fourier sine series:

\[ \delta x(t) = \sum_{n=1}^{\infty} b_n \sin(n\omega t) \]  

(5)

with \( \omega = 2\pi/T \). By considering all possible values of \( b_n \), we consider all possible paths between these two points in space-time. We can compute the action for \( x(t) \):

\[ S = \int_0^T dt \left( \frac{1}{2} m \dot{x}(t)^2 + \sum b_n n\omega \cos(n\omega t) \right)^2 \]  

\[ = \frac{1}{2} mv^2 T + \frac{m}{2} \sum_{n, n'} \omega^2 b_n b_{n'} \int dt \cos(n\omega t) \cos(n'\omega t). \]  

The integral is \( \frac{T}{2} \delta_{nn'} \), so we have

\[ S = S_{cl} + \frac{m}{4} T \omega^2 \sum n^2 b_n^2. \]  

(7)

This is clearly minimized if all the \( b_n \)'s are zero, i.e. if \( x = x_{cl} \).