Physics 171. General Relativity. Professor Dine

Fall, 2009. Handout: The Action Principle

Consider a particle in one dimension, with a lagrangian:

$$L = \frac{1}{2}m\dot{x}^2 - V(x). \tag{1}$$

We have seen that the action principle gives a differential equation for x, which is just Newton's equation. But it is not hard, in the case of a free particle, to actually compute the action for all possible paths, and verify that the classical solution gives the minimum value of the action.

Suppose we have a particle which starts at time $t_1 = 0$ at $x_1 = 0$, and at time T sits at $x_2 = vT$. Then the classical path is:

$$x_{cl}(t) = vt. \tag{2}$$

Now we want to consider some other path. This can be written as

$$x(t) = x_{cl}(t) + \delta x(t). \tag{3}$$

Because we specify the initial and final position of the particle, and x_{cl} satisfies these conditions, we have:

$$\delta x(0) = \delta x(T) = 0. \tag{4}$$

So we can expand $\delta x(t)$ in a Fourier sine series:

$$\delta x(t) = \sum_{n=1}^{\infty} b_n \sin(n\omega t) \tag{5}$$

with $\omega = 2\pi/T$. *By considering all possible values of b_n, we consider all possible paths between these two points in space-time.* We can compute the action for $x(t)$:

$$S = \int_0^T dt \frac{1}{2}m(x_{cl}(t) + \sum b_n n\omega \cos(n\omega t))^2 \tag{6}$$

$$= \frac{1}{2}mv^2T + \frac{m}{2}\sum_{n,n'} \omega^2 b_n n b_{n'} n' \int dt\, \cos(n\omega t)\cos(n'\omega t).$$

The integral is $\frac{T}{2}\delta_{nn'}$, so we have

$$S = S_{cl} + \frac{m}{4}T\omega^2 \sum n^2 b_n^2. \tag{7}$$

This is clearly minimized if all the b_n's are zero, i.e. if $x = x_{cl}$.