Covering the black hole space-time

The singularity in the Schwarzschild metric is a coordinate singularity. We can see this by going to new coordinates ("Eddington - Finkelstein")

\[ t = \varphi - r - 2M \ln \left| \frac{\varphi}{2M} - 1 \right| \]

\[ = - \left(1 - \frac{2M}{r} \right) d\varphi^2 + 2d\varphi dr + r^2 (d\phi^2 + \sin^2 \phi d\phi^2) \]

(study near the Schwarzschild radius)

\[ r = 2M + x = 2M \left(1 + \frac{x}{2M} \right) \quad dr = dx \]
\[ t = \varphi - r - 2M \ln \left(1 + \frac{x}{2M} \right) \quad dt = d\varphi - dr - \frac{2M}{x} dx \]

\[ ds^2 = -dt^2 \left(1 - \frac{2M}{r} \right)^{-1} + dr^2 \left(1 - \frac{2M}{r} \right)^{-1} + r^2 d\Omega^2 \]

\[ = -d\varphi^2 \left(1 + \frac{2M}{x} \right) + \left(1 + \frac{2M}{x} \right) 2d\varphi dx + \frac{x}{2M} \frac{4M}{x} d\varphi dx \]
\[ - \frac{4M^2}{x^2} dx^2 \left(\frac{1}{2M}\right) + dx^2 \left(\frac{2M}{x}\right) \]
\[ = -\frac{x}{2M} d\varphi^2 + 2d\varphi dx + 4M^2 dx^2 \]
Light cones in the EF coordinates

\[ d\theta = d\phi = 0 \]

\[- (1 - \frac{2M}{r}) dv^2 + 2dvdr = 0 \]

Solsn:

\[ u = \text{const} \quad \text{(large } r: \ t + r = \text{const} \Rightarrow \text{radially infalling photon}) \]

\[ \frac{dv}{dr} = 2 \left( 1 - \frac{2M}{r} \right)^{-1} = \frac{2}{1 - \frac{2M}{r}} = 2 + \frac{4M}{r-2M} \]

\[ u = 2 \left( r + 2M \ln \left| \frac{r}{2M} - 1 \right| \right) + \text{const} \]

\[ r > 2M \text{ but close to } 2M: \]

\[ u \rightarrow -\infty \]

\[ t = u - r - 2M \ln u = 2M \ln \left( \frac{r}{2M} - 1 \right) \rightarrow -\infty \]

\[ \Rightarrow \text{outgoing light rays} \]
\[ r < 2M \quad r \to 2M \]

Again, \( t \to -\infty \).

Only infalling light rays for \( r < 2M \).

Red shift from collapsing stars.

We have seen objects reach singularity in finite proper time, but infinite Schwarzschild time.

We can reconcile these facts by imagining the infalling observer sending out "help" messages at (very short) intervals in his proper time, and asking what these look like to a far away observer.
Consider receiver, emitter coordinates

\[ (r_R, t_R) \quad (u_R, t_R) \]

\[ r_R \gg r, \quad t_R \gg r_R, \quad u_R \approx t_R + r_R \]

\[ (t_E, t_E) \quad (u_E, t_E) \]

All of the coordinates lie on the path of an outgoing light ray,

\[ u = 2(r + 2M \ln |r_{2M} - 1|) + \text{const} \]

\[ t = u - r - 2M \ln \left| \frac{c}{2M} - 1 \right| \]

As \( r = r_E \to 2M \), log dominates

\[ \frac{u_E}{2M} - u_R \approx -2r_R + 4M \ln \left( \frac{u_E}{2M} - 1 \right) \]

\[ u_E \approx \frac{4M}{2M} \ln \left( \frac{u_E - 1}{r_R} \right) \]

\[ t_R \approx -4M \ln \left( \frac{u_E}{2M} - 1 \right) \quad \text{\( (t_R \gg r_R) \)} \]

\[ \frac{r_E}{2M} - 1 = e^{-(t_R - r_R)/4M} \quad \text{\( (u_E \text{ odd}) \)} \]
So if in interval $\Delta t_j$ travel $\Delta r_E$, 

$$\Delta t_{12} \approx \frac{\Delta r_E}{|r_{12}|^{\frac{1}{2m}} - 1}$$

$\Rightarrow$ longer and longer time between reception of signals.

Corresponding redshifts signals weaker and weaker; object disappears.

Can show (see Hartle)

$$w_E \sim e^{-tr/I_4m} w_E$$
Kruskal Coordinates

\[ ds^2 = -dt^2 \left(1 - \frac{2m}{r}\right) + dr^2 \left(1 - \frac{2m}{r}\right)^{-1} + r^2 d\Omega^2 \]

\[ U = \left(\frac{r}{2m} - 1\right)^{\frac{1}{2}} e^{r/4m} \cosh\left(\frac{t}{4m}\right) \]

\[ V = \left(\frac{r}{2m} - 1\right)^{\frac{1}{2}} e^{r/4m} \sinh\left(\frac{t}{4m}\right) \]

\[ dU = \left(\frac{r}{2m} - 1\right)^{\frac{1}{2}} e^{r/4m} \sinh\left(\frac{r}{4m}\right) \frac{dt}{4m} + \left[\frac{1}{4m} \left(\frac{r}{2m} - 1\right)^{\frac{1}{2}} e^{r/4m} \cosh\left(\frac{r}{4m}\right) \right] \frac{dr}{2m} \]

\[ dV = \ldots \cosh + \ldots \sinh \]

\[ ds^2 = \frac{32m^2}{r} e^{-r/2m} \left(-dv^2 + du^2\right) + r^2 (u, v) \ d\Omega^2 \]

Special places: \( \left(\frac{r}{2m} < 1; \text{ reverse sign }\right) \)

\[ r = 0 \quad U = \cosh\left(\frac{t}{4m}\right) \quad V = \sinh\left(\frac{t}{4m}\right) \]

\[ U^2 - V^2 = 1 \quad (\text{hyperbola}) \]

\[ r = r_3 = 2m: \quad U = V = 0 \quad (t \neq 0) \]

Light cones: \[ U = \pm V + c \]
Enlarge by noting:
\[(r^{\frac{3}{2}m} - 1)e^{r^{\frac{3}{2}m}} = U^2 - V^2\]

\[U^2 - V^2 = \text{const}; \text{ hyperbolas}\]

\[r = 2m: \quad U = \pm V\]

But also:
\[t = \pm \infty: \quad V = U\]
\[t = -\infty: \quad V = -U\]

So away from \(r = 0\), \(U = V\) so \(t = \pm \infty, r > r_s\)
in Schwarzschild.

\[
\begin{align*}
\text{timelike} \\
\text{world line}
\end{align*}
\]
7.1 Coordinates

BOX 7.1 The Penrose Diagram for Flat Space

Another example of a useful coordinate system for flat space is the one used to construct its Penrose diagram. Begin with the line element for flat spacetime in spherical polar coordinates (7.4). Replace \( t \) and \( r \) by two new coordinates \( u \) and \( v \) defined by

\[
    u = t - r, \quad v = t + r
\]
so that the line element becomes

\[
ds^2 = -du\,dv + \frac{1}{4}(u-v)^2(d\theta^2 + \sin^2 \theta\,d\phi^2).\
\]

The \((u, v)\) axes are rotated with respect to the \((t, r)\) axes by 45°, as shown in the \((t, r)\) spacetime diagram. Radial light rays travel on lines of constant \( u \) or constant \( v \). That is evident either from the definitions of these coordinates in (a) or because (b) shows that lines of constant \( \theta \), \( \phi \), and either \( u \) or \( v \) have \( ds^2 = 0 \).

![Penrose Diagram](http://example.com/penrose.png)

By this mapping of infinity to finite coordinate values, it is possible to distinguish different kinds of infinity. Outgoing radial light rays—with \( t = r + \text{constant} \)—are lines of constant \( u' \). They wind up on the boundary \( u' = \pi/2 \), called future null infinity and denoted by \( S_+ \). Particle trajectories that lie within the local light cone start from the point \((t' = -\infty, r' = 0)\), called past null infinity, \( I_- \), and wind up at the point \((t' = +\infty, r' = 0)\), called future timelike infinity, \( I_+ \). (Problem 4). Similarly, infinite spacelike curves wind up at the point \( I_0 \), which labels a sphere called spacelike infinity.

Make a further transformation of \( u \) and \( v \) to new coordinates \( u' \) and \( v' \) and corresponding new coordinates \( t' \) and \( r' \) with the relations:

\[
    u' = \tan^{-1} u \equiv t' - r', \quad v' = \tan^{-1} v \equiv t' + r'.
\]

The \( t \) and \( r \) coordinates for flat spacetime have the ranges \(-\infty < t < +\infty, 0 < r < +\infty \). But the new coordinates \( t' \) and \( r' \) lie between \(-\pi/2 \) and \( +\pi/2 \), so the ranges for \((t', r')\) are finite. In fact, all the \((t, r)\) plane of flat spacetime is mapped into the finite region \( r' > 0,\ t' > -\pi/2,\ t' > +\pi/2 \) shown lightly shaded in the \((t', r')\) plane at top right. This is the Penrose diagram for flat spacetime.

![Penrose Diagram](http://example.com/penrose2.png)
timelike and null world lines lead to the singularity at $r = 0$, demonstrating its inevitable formation once the star's surface crosses the Schwarzschild radius. No light rays or timelike world lines escape from the inside of the horizon, and events there remain hidden from any observer outside.

The world lines of the two communicating observers discussed in Section 12.2 and illustrated in Figure 12.3 are shown in the Kruskal diagram in Figure 12.7. The distant observer runs along a hyperbola of large, fixed $r$. The light rays emitted at equal proper time intervals by the falling observer move on the dotted 45° lines shown. They evidently are received less and less frequently at later and later times by the distant observer leading the increasing redshift of the light from the collapsing star and the extinction of its luminosity. The last light ray received is emitted just before the star and falling observer plunge through the Schwarzschild radius.

**Box 12.5 The Penrose Diagram for the Schwarzschild Geometry**

By a careful choice of new coordinates $(U', V')$, it is possible to relabel the points of a Kruskal diagram so that light rays continue to propagate along 45° lines in the new coordinates such that points at infinity are labeled by finite coordinate values rather than infinite ones. The resulting picture of the whole slice of the Kruskal extension (Box 12.4) of the Schwarzschild geometry in a finite region of the $(U', V')$ plane is called the Penrose diagram for the Schwarzschild geometry and is a useful tool for picturing its global spacetime structure. The construction of a Penrose diagram for flat spacetime was described in Box 7.1 on p. 137, and the construction for the Schwarzschild geometry is closely parallel. Begin with the Schwarzschild geometry in Kruskal–Szekeres coordinates (12.14) and replace coordinates $U$ and $V$ with two new coordinates $u$ and $v$ defined by

$$U = (v - u)/2, \quad V = (v + u)/2. \quad (a)$$

The $uv$ axes are just the $UV$ axes rotated by 45° so that light rays move on curves of constant $u$ or $v$. Introduce other coordinates $(u', v')$ and $(U', V')$ defined by

$$u' = \tan^{-1}(u) = V' - U', \quad v' = \tan^{-1}(v) = V' + U'. \quad (b)$$

Light rays move on curves of constant $u'$ and $v'$, i.e., the 45° lines in the $U'V'$ plane. The infinite ranges of $u$ and $v$ are each mapped into the finite range $(-\pi/2, \pi/2)$ for $u'$ and $v'$. With a little work one sees that the hyperbola $r = 0$, $V > 0$ maps into the line $V' = \pi/4$, $-\pi/4 < U' < \pi/4$, whereas the one with $V < 0$ maps into the same line at $V' = -\pi/4$. The horizon $V = U$ maps into the same 45° line in the $U'V'$ plane. The resulting Penrose diagram is shown at left. As in flat space, it is possible to identify different kinds of infinity: future and past null infinity $\mathcal{I}_{\pm}$, where light rays at infinity start and wind up, future and past timelike infinity $\mathcal{I}_{\pm}$, where timelike world lines start and wind up, and spacelike infinity $I_0$, which all spacelike surfaces at infinity intersect. There are two sets of these, one for each asymptotic region. With this diagram we can see that the horizon is the boundary of the region of spacetime that can be connected to future null infinity by a light ray.

Don't mix up this $v$ with the Eddington–Finkelstein $\nu$ coordinate!