Physics 171. General Relativity. Professor Dine

Fall, 2009. Handout: Some Notes on Geometry, Differentials

It is worth going over a few points from the lecture.

1 Local Cartesian Coordinates

In general relativity, it will be important that locally, over a sufficiently small region, we can always set up coordinates corresponding to a flat space. We discussed briefly how this works for the sphere. In a bit more detail, consider the case of the three-sphere, and study the system near $\phi = 0$, $\theta = \pi/2$. Then, from

$$z = R\cos\theta \quad x = R\sin\theta\cos\phi \quad y = R\sin\theta\sin\phi \tag{1}$$

we have

$$dz = -R\sin\theta d\theta \approx -Rd\theta \tag{2}$$

$$dx = R\cos\theta d\theta\cos\phi - R\sin\theta\sin\phi d\phi \approx 0 \quad dy = R\cos\theta d\theta\sin\theta + R\sin\theta\cos\phi d\phi \approx Rd\phi.$$

So only dz and dy are non-zero, i.e. we have two Cartesian coordinates;

$$ds^2 = dz^2 + dy^2 + dz^2 \approx dz^2 + dy^2 \approx R^2(d\theta^2 + d\phi^2). \tag{3}$$

We could keep higher order terms in the expansions of $\sin\theta$ and $\cos\theta$. Then we would get corrections to ds^2 in powers of $x^2/r^2, y^2/R^2$ (e.g. $y \approx \phi R$). So as R gets large, the space looks more and more flat.

2 Stereographic projection

This is a mapping of the sphere onto the plane. The coordinates of the plane, x and y, are related to those of the sphere, x_1, x_2, x_3, $x_i^2 = 1$, by

$$x = \frac{x_1}{1 - x_3} \qquad y = \frac{x_2}{1 - x_3}. \tag{4}$$

Under the mapping, the equator, $x_3 = 0$, is mapped into the unit circle on the plane. The southern hemisphere is mapped into the interior of the circle; the northern hemisphere is mapped to the exterior. If we want our map to be a circle of finite size, we need to cut out all of the sphere above some specified longitude.

Let's check what ds^2 looks like in terms of x and y, i.e. how we make a correspondence between distances on the sphere and distances on the plane. We claim

$$ds^2 = 4\frac{dx^2 + dy^2}{(1 + x^2 + y^2)^2} \tag{5}$$

Let's verify the claim. Start with

$$dx = \frac{dx_1(1 - x_3) + x_3 dx_3}{(1 - x_3)^2}. \quad dy = \frac{dx_2(1 - x_3) + x_3 dx_3}{(1 - x_3)^2}. \tag{6}$$

So

$$dx^2+dy^2 = \frac{dx_1^2(1-x_3)^2 + x_1^2dx_3^2 + dx_2^2(1-x_3)^2 + x_2^2dx_3^2}{(1-x_3)^4} + \frac{2dx_1dx_3x_1(1-x_3) + 2dx_2dx_3x_2(1-x_3)}{(1-x_3)^4} \tag{7}$$

We can simplify the numerator. Note, first, that since

$$x_1^2 + x_2^2 + x_3^2 = 1 \tag{8}$$

we have

$$x_1dx_1 + x_2dx_2 = -x_3dx_3 \tag{9}$$

(this is the statement that, since the points x_i are constrained to lie on the sphere, they can't all be varied independently). So we have

$$dx^2 + dy^2 = \frac{(dx_1^2 + dx_2^2)(1-x_3)^2 + (x_1^2 + x_2^2)dx_3^2 - 2dx_3^2(1-x_3)x_3}{(1-x_3)^4} \tag{10}$$

$$= \frac{(dx_1^2 + dx_2^2)(1-x_3)^2 + dx_3^2(1 + x_3^2 - 2x_3)}{(1-x_3)^4}$$

$$= \frac{(dx_1^2 + dx_2^3 + dx_3^2)}{(1-x_3)^2}.$$

Finally, note that, substituting for x and y their expressions in terms of x_i:

$$\frac{1}{1+x^2+y^2} = \frac{(1-x_3)^2}{(1-x_3)^2 + x_1^2 + x_2^2} = \frac{(1-x_3)^2}{(1 + x_ix_i - 2x_3)} = \frac{(1-x_3)^2}{2(1-x_3)}. \tag{11}$$

So plugging in our expression for ds^2, we see that

$$ds^2 = dx_idx_i. \tag{12}$$

I STRONGLY URGE YOU TO VERIFY THE STEPS IN THIS DERIVATION. I WON'T MAKE YOU HAND THIS IN, BUT I CLAIM THAT IF YOU DO THIS, YOU'LL NEVER HAVE ANY DISCOMFORT WITH DIFFERENTIALS AGAIN.