It is worth going over a few points from the lecture.

1 Local Cartesian Coordinates

In general relativity, it will be important that locally, over a sufficiently small region, we can always set up coordinates corresponding to a flat space. We discussed briefly how this works for the sphere. In a bit more detail, consider the case of the three-sphere, and study the system near $\phi = 0, \theta = \pi/2$. Then, from

$$ z = R \cos \theta \quad x = R \sin \theta \cos \phi \quad y = R \sin \theta \sin \phi $$

we have

$$ dz = -R \sin \theta d\theta \approx -R d\theta \quad (2) $$
$$ dx = R \cos \theta d\theta \cos \phi - R \sin \theta \sin \phi d\phi \approx 0 \quad dy = R \cos \theta d\theta \sin \theta + R \sin \theta \cos \phi d\phi \approx R d\phi. $$

So only $dz$ and $dy$ are non-zero, i.e. we have two Cartesian coordinates;

$$ ds^2 = dz^2 + dy^2 \approx dz^2 + dy^2 \approx R^2 (d\theta^2 + d\phi^2). $$

We could keep higher order terms in the expansions of $\sin \theta$ and $\cos \theta$. Then we would get corrections to $ds^2$ in powers of $x^2/r^2, y^2/R^2$ (e.g. $y \approx \phi R$). So as $R$ gets large, the space looks more and more flat.

2 Stereographic projection

This is a mapping of the sphere onto the plane. The coordinates of the plane, $x$ and $y$, are related to those of the sphere, $x_1, x_2, x_3, x_i^2 = 1$, by

$$ x = \frac{x_1}{1-x_3} \quad y = \frac{x_2}{1-x_3}. $$

Under the mapping, the equator, $x_3 = 0$, is mapped into the unit circle on the plane. The southern hemisphere is mapped into the interior of the circle; the northern hemisphere is mapped to the exterior. If we want our map to be a circle of finite size, we need to cut out all of the sphere above some specified longitude. Let’s check what $ds^2$ looks like in terms of $x$ and $y$, i.e. how we make a correspondence between distances on the sphere and distances on the plane. We claim

$$ ds^2 = 4 \frac{dx^2 + dy^2}{(1 + x^2 + y^2)^2} $$

Let’s verify the claim. Start with

$$ dx = \frac{dx_1(1-x_3) + x_3 dx_3}{(1-x_3)^2}, \quad dy = \frac{dx_2(1-x_3) + x_3 dx_3}{(1-x_3)^2}. $$
So
\[ dx^2 + dy^2 = \frac{dx_1^2(1-x_3)^2 + x_1^2dx_3^2 + dx_3^2(1-x_3)^2 + x_2^2dx_3^2 + 2dx_1dx_3x_1(1-x_3) + 2dx_2dx_3x_2(1-x_3)}{(1-x_3)^4} \] (7)

We can simplify the numerator. Note, first, that since
\[ x_1^2 + x_2^2 + x_3^2 = 1 \] (8)
we have
\[ x_1dx_1 + x_2dx_2 = -x_3dx_3 \] (9)
(this is the statement that, since the points \( x_i \) are constrained to lie on the sphere, they can’t all be varied independently). So we have
\[ dx^2 + dy^2 = \frac{(dx_1^2 + dx_2^2)(1-x_3)^2 + (x_1^2 + x_2^2)dx_3^2 - 2dx_3^2(1-x_3)x_3}{(1-x_3)^4} \] (10)
\[ = \frac{(dx_1^2 + dx_2^2)(1-x_3)^2 + dx_3^2(1 + x_3^2 - 2x_3)}{(1-x_3)^4} \]
\[ = \frac{(dx_1^2 + dx_2^2 + dx_3^2)}{(1-x_3)^2}. \]

Finally, note that, substituting for \( x \) and \( y \) their expressions in terms of \( x_i \):
\[ \frac{1}{1 + x_3^2 + y_3^2} = \frac{(1-x_3)^2}{(1-x_3)^2 + x_1^2 + x_2^2} = \frac{(1-x_3)^2}{(1 + x_1x_i - 2x_3)} = \frac{(1-x_3)^2}{2(1-x_3)}. \] (11)

So plugging in our expression for \( ds^2 \), we see that
\[ ds^2 = dx_idx_i. \] (12)

I STRONGLY URGE YOU TO VERIFY THE STEPS IN THIS DERIVATION. I WON’T MAKE YOU HAND THIS IN, BUT I CLAIM THAT IF YOU DO THIS, YOU’LL NEVER HAVE ANY DISCOMFORT WITH DIFFERENTIALS AGAIN.