Problem numbers refer to your textbook.

1. 12-5.

**Solution:**
We want to look inside the Schwarschild radius, so we can use Eddington-Finkelstein coordinates. Then

\[ ds^2 = -(1 - \frac{2M}{r})dv^2 + 2dvdr \]

The relevant Killing vector is:

\[ \xi^v = (1, 0, 0, 0) \]

So

\[ u \cdot \xi^v = -(1 - \frac{2M}{r}) \frac{dv}{d\tau} + \frac{dr}{d\tau} \equiv e \]

So

\[ \frac{dv}{d\tau} = \frac{e + \frac{dr}{d\tau}}{(1 - 2M/r)} \]

So

\[ u^\mu u^\nu g_{\mu\nu} = -1 \]

gives

\[ -1 = -(1 - 2M/r)^{-1} \left( \frac{dv}{d\tau} \right)^2 + 2 \frac{dv}{d\tau} \frac{dr}{d\tau}. \]

Solving for \( \frac{dv}{d\tau} \) gives

\[ \frac{dr}{d\tau} = \left[ -(1 - \frac{2M}{r}) + e^2 \right]^{1/2} \]

From \( \frac{dr}{d\tau} = 0 \) at \( r = 10 \), we can determine \( e \); then we can integrate to solve for \( \tau \).

\[ e^2 = (1 - 2M/10) \]

So we obtain:

\[ \tau = \int_0^{10} \frac{dr}{\sqrt{2(-1/10 + M/r)}} \]

\[ = 5\sqrt{5}\pi M \]

2. 12-8.

**Solution:** The Penrose diagram for the black hole is particularly useful for this. Just consider a vertical trajectory, and ask which 45° lines intersect it. Some do; some don’t.
3. 12-12.

**Solution:** In the Eddington-Finkelstein coordinates, it is easy to see what the tangents are: changing $v$ or $\theta$ or $\phi$ does not take you out of the Schwarschild horizon. So the tangent vectors are (where the components of vectors are $(v, r, \theta, \phi)$):

$$(1, 0, 0, 0); (0, 0, 1, 0); (0, 0, 0, 1)$$

and the normal vector is

$$\mathbf{n} = (1, 0, 0, 0)$$

(note that because of the off-diagonal structure of the metric, and the fact that the coefficient of $dv^2$ vanishes on the horizon, $\mathbf{n} \cdot \mathbf{n} = 0$, but it is not orthogonal to the others). (I haven’t worried about the normalization of any of these). But in the EF metric, $\mathbf{n}$ is a null vector, since $g_{rr} = 0$.

4. 12-23.

**Solution:** In Kruskal coordinates, the light cones are simple:

$$V = U + C$$

Now for large $r_r$, $t_r$, we can write:

$$U_r \sim e^{r/4M} e^{r/4M} + e^{-t/4M} e^{-t/4M}$$

$$V_r \sim e^{r/4M} e^{r/4M} - e^{-t/4M} e^{-t/4M}$$

Now the issue is to determine the constant $c$. For $t \sim r$, the constant $C$ is of order one. But as we keep $r$ fixed and let $t$ get large, the constant $C$ tends to zero, as $e^{-t/4M}$. Now consider the emission point. Near the emission point, we can simplify things by taking $t$ small. So

$$U_E \approx (r_E/2m - 1)^{1/2} e^1$$

$$V_E \approx 0.$$  

Since

$$V_E - V_R = U_E - U_R$$

we have, with these approximations:

$$e^{(r_R-t_R)/4M} \approx (r_E/2M - 1)^{1/2}.$$  

(Note that, as in my class lectures, this differs from your text by a factor of two in the exponent).

The following problems are from the curvature handout. These are mostly straightforward extensions of the text. Exercise 6 is largely solved in the added notes to the curvature handout. I will provide more notes on these if I see a need as I grade the problems.

5. Exercise 1.

**Solution:**

This is simply an application of the chain rule:

$$\frac{\partial f}{\partial x^\mu} = \frac{\partial x^\nu}{\partial x^\mu} \frac{\partial f}{\partial x^\nu}.$$  

For the verification that the product of $g$’s is invariant, need to use the fact that

$$\frac{\partial x^\rho}{\partial x^\mu} \frac{\partial x^\mu}{\partial x^\sigma} = \frac{\partial x^\rho}{\partial x^\sigma} = \delta^\rho_\sigma$$
6. Exercise 2

Solution:

This is just a little bit of algebra beyond what appears in the lecture notes.

7. Exercise 3.

Solution:

The metric in polar coordinates is

\[ ds^2 = d\rho^2 + \rho^2 d\phi^2. \]

The non-vanishing Christoffel symbols are:

\[ \Gamma^\phi_{\phi\rho} = \frac{1}{2} g^{\phi\phi} (\partial_\rho g_{\phi\phi} + \text{vanishing}) = \frac{1}{\rho}; \quad \Gamma^\rho_{\phi\phi} = \frac{1}{2} g^{\rho\rho} (\text{vanishing} - \partial_\rho g_{\phi\phi}) = -\rho. \]

Here we have noted the diagonal form of the metric and have avoided writing down many terms which clearly vanish. From these we can construct the various components of the covariant derivative of a vector. In general,

\[ D_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho. \]

So the individual components are, again noting that many components of \( \Gamma \) vanish:

\[ D_\rho V^\rho = \partial_\rho V^\rho \quad D_\rho V^\phi = \partial_\rho V^\phi + \Gamma^\phi_{\rho\phi} V^\phi. \]

\[ D_\phi V^\rho = \partial_\phi V^\rho + \Gamma^\rho_{\phi\rho} V^\phi \quad D_\phi V^\phi = \partial_\phi V^\phi + \Gamma^\phi_{\phi\rho} V^\rho. \]

In particular, this means:

\[ D_\mu V^\mu = \partial_\rho V^\rho + \partial_\phi V^\phi + \frac{1}{\rho} V^\phi. \]

If \( V^\mu = \partial^\mu f \), we can write the laplacian as (remember to contract the indices with the metric or inverse metric):

\[ \partial^2 f = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{1}{\rho} \frac{\partial f}{\partial \rho}. \]

You can check these formulas against the inner cover of Griffiths E&M text; clearly, with slightly more work, you can use these techniques to derive all of the other formulas there.


Solution:

This is just a little bit of algebra beyond what appears in the lecture notes.


Solution:

This is just a little bit of algebra beyond what appears in the lecture notes.

Solution:

This is largely done in the notes that I added to the lectures. The case of a cosmological constant follows from the Friedman equation:

\[ \dot{a}^2 - \frac{8\pi G \rho}{3} a^2 = 0. \]  \hspace{1cm} (1)

For this case, \( \rho = \Lambda \) is a constant, and so \( a \) grows exponentially.