Physics 171. General Relativity. Professor Dine

Fall, 2009. Handout: Special Relativity and Four Vectors

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1 Time Dilation

1. Time Dilation: From Lorentz transformations:

$$t'_1 - t'_2 = \gamma(t_1 - t_2).$$

   But this is more general. One can consider a frame *instantaneously* at rest with respect to a particle, which might not be in uniform motion; the frame has velocity $v'(t')$ with respect to some inertial frame. If t is the time in the rest frame, then

$$d\tau^2 = dt^2 = dt'^2 - d\vec{x}^2 = dt'^2 \left(1 - \left(\frac{dx'^2}{dt'^2}\right)\right) = dt'^2(1 - v'^2). \tag{1}$$

   Note that not only do we see that the time dilation is quite general, but also that there was no special choice of axes.

   From this point of view, we can readily understand the twin paradox. This is the possibility of sending one's twin off in a rocket at speeds near the speed of light, who returns after some years. The twin is much younger (by the integral of the time dilation factor over the trajectory). The potential paradox is in the asymmetry between the twins, but because the travelling twin was not in an inertial frame, she has experienced acceleration, while her sister has not.

2. Addition of velocities: the Galilean rule is replaced by (using my usual rule that differentials can be divided and multiplied like ordinary numbers):

$$V^{x\prime} = \frac{dx + vdt}{dt + vdx} = \frac{\frac{dx}{dt} + v}{1 + vV^x} \tag{2}$$

$$= \frac{V^x + v}{1 + 1 + vV^x/c^2}$$

   (I have used dimensional analysis to restore the factors of c). Similarly

$$V^{y\prime} = \frac{V^y}{1 - vV^x/c^2}\frac{1}{\gamma} \tag{3}$$

   and similarly for $V^{z\prime}$. Note that if $V^x = c$, $V^{x\prime} = \frac{c+v}{1+v/c} = c$.

2 The Mathematics of Special Relativity

Three vectors: abstract idea of vector represented concretely in many ways:

$$\vec{x} = (x, y, z) = (x^1, x^2, x^3) = \{x^i\}$$

Objects which transform under rotations like $\vec{x}$ are, by definition, vectors. Familiar examples:

$$\vec{v} = \frac{d\vec{X}}{dt} \quad \vec{a} = \frac{d^2\vec{x}}{dt^2} \quad \vec{p} = m\vec{v}$$

Vector equations:

$$\vec{F} = m\vec{a} = -G\vec{\nabla}\Phi(\vec{x} - \vec{X}).$$

All sides of these equations transform the same way under rotations, so true in any frame.

In space-time, four-vectors. Can again think of abstract vectors, and represent in various ways. We will use bold face, as in your text, to indicate an abstract four vector (also, as in the text, on the blackboard and in handwriting, we'll use an underscore)

$$\mathbf{x} = (t, x, y, z) = x^\alpha, \alpha = 0, 1, 2, 3 = t, x, y, z. \tag{4}$$

Can add, pictorially, in a space-time diagram as for ordinary vectors.

Length of a four vector:

$$s^2 = \mathbf{x^2} = \vec{x}^2 - \vec{x}^{0\ 2} \tag{5}$$

Constructing other four vectors: start with dx^μ. Use the fact that $d\tau$ is a scalar to define the analog of the ordinary velocity, the four velocity:

$$u^\mu = \frac{dx^\mu}{d\tau}. \tag{6}$$

Remember that $d\tau = \gamma^{-1}dt$, so

$$\frac{dx^0}{d\tau} = \gamma; \frac{dx^i}{d\tau} = \gamma v^i. \tag{7}$$

An interesting feature of the four velocity is that

$$\mathbf{u}^2 = u^{i\ 2} - u^{0\ 2} = \gamma^2(v^2 - 1) = -1. \tag{8}$$

Perhaps more interesting is the four-momentum:

$$p^\mu = mu^\mu \tag{9}$$

with components

$$p^0 = m\gamma = E; \vec{p} = mv^i\gamma \tag{10}$$

The identification with the energy is natural, particularly if we look at the limit of small velocities:

$$p^0 = mc^2 + \frac{1}{2}mv^2. \tag{11}$$

***** You should check this***** Note that

$$\mathbf{p^2} = m^2\mathbf{u^2} = -m^2. \tag{12}$$

This is the relation:

$$E^2 = p^2c^2 + m^2c^4. \tag{13}$$

2.1 Basis vectors, dot products

Ordinary vector: components, basis vectors:

$$\vec{x} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$= x\hat{i} + y\hat{j} + z\hat{k}$$

etc. We might write this as:

$$\vec{x} = \sum x^i \hat{e}_i = x^i \hat{e}_i$$

(Einstein summation convention again).

Four vectors: add one more basis vector, $\hat{e}_0$ or $\hat{e}_t$:

$$x^\mu = (x^0, x^i); \ \mathbf{x} = x^\mu e_\mu$$

For ordinary vectors: dot product:

$$\vec{x} \cdot \vec{y} = x^i y^i$$

Important because a scalar; takes same value if rotate coordinate system. In terms of basis vectors:

$$\vec{x} \cdot \vec{y} = x^i y^i \hat{e}_i \cdot \hat{e}_j = x^i x^j \delta_{ij} = x^i y^i.$$

For four vectors:

$$\mathbf{x} \cdot \mathbf{y} = \vec{x}^2 - x^{0\ 2}.$$

This is Lorentz invariant. For the dot products of the basis vectors, this works provided:

$$\vec{e}_i \cdot \vec{e}_j = \delta_{ij}; e_0 \cdot e_0 = -1 \tag{14}$$

which can be summarized:

$$\mathbf{e}_\mu \mathbf{e}_\nu = \eta_{\mu\nu} \tag{15}$$

$\eta_{\mu\nu}$ is called the metric of flat spacetime;

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{16}$$

We can also write:

$$\mathbf{x} \cdot \mathbf{y} = x^\mu x^\nu \eta_{\mu\nu}. \tag{17}$$

We can define raising and lowering of indices:

$$x_\mu = \eta_{\mu\nu} x^\nu \tag{18}$$

so

$$\mathbf{x} \cdot \mathbf{y} = x^\mu y_\mu \tag{19}$$

Similarly, $y^\mu = \eta^{\mu\nu} y_\nu$, raises the indices.

In terms of components:

$$x^\mu = (t, \vec{x}) \qquad x_\mu = (-t, \vec{x}). \tag{20}$$

***** You should practice a few times raising, lowering indices on vectors; writing dot products in these various ways*****

3 Covariant equations

For a free particles, equations of motion can be written in a Lorentz-covariant way (compare, again, vector equations):

$$\frac{du^\mu}{d\tau} = 0 \tag{21}$$

Note that this is four equations. These have the same content, though, as Newton's three equations. The $\mu = 0$ equation says that γ is a constant. The other three then say the v^i is constant. Clearly there is redundancy. This comes from the fact that, as we saw, $\mathbf{u} \cdot \mathbf{u} = -1$, so

$$\frac{d}{d\tau}(u^\mu u_\mu) = 2\frac{du^\mu}{d\tau} u_\mu = 0. \tag{22}$$

In the presence of forces, this must become an equation of the form

$$\frac{du^\mu}{d\tau} = f^\mu. \tag{23}$$

In 1905, Einstein knew about one Lorentz-covariant system, electromagnetism. If this is to hold together, it should be possible to write the Lorentz force law in this way. We won't show now that this is a four-vector, but the four force has components:

$$f^0 = q\vec{u} \cdot \vec{E} \quad \vec{f} = qu^0\vec{E} + q\vec{u} \times \vec{B}. \tag{24}$$

Non-relativistically, the spatial components of eqn. 23 reproduce the Lorentz force law. The time component expresses the change of energy of a particle in an electromagnetic field:

$$\frac{dE}{dt} = q\vec{v} \cdot \vec{E}. \tag{25}$$

***** You should check that with this identification of the components of the four-force, we reproduce the Lorentz force law.*****

4 The Variational Principle

The motion of a particle, it turns out, extremizes:

$$\int_A^B d\tau.$$

A simple-minded verification comes from introducing a constant, and writing

$$S = -m \int d\tau = -m \int \sqrt{dt^2 - d\vec{x}^2} \tag{26}$$

$$= -m \int dt \sqrt{1 - v^2}$$

$$\approx \int dt(mc^2 + \frac{1}{2}mv^2).$$

Up to a constant, this is the usual action for a free particle.

We can also derive the covariant equation of motion:

$$S = -m \int \left(\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau}\right)^{1/2}. \tag{27}$$

Now do the usual variational thing

$$x^\mu \to x^\mu + \delta x^\mu \tag{28}$$

(where, by assumption, x^μ solves the equation of motion). Substitute in the action, and demand that the term linear in δx^μ vanishes (action is stationary):

$$\begin{aligned} S &= -m \int d\tau \left[\left(\frac{dx^\mu}{d\tau} + \frac{d\delta x^\mu}{d\tau} \right) \left(\frac{dx_\mu}{d\tau} + \frac{d\delta x_\mu}{d\tau} \right) \right]^{1/2} \\ &= -m d\tau \left[\left(\frac{dx^\mu}{d\tau} \frac{dx_\mu}{d\tau} + 2 \frac{d\delta x^\mu}{d\tau} \frac{dx_\mu}{d\tau} \right) \right] \end{aligned} \tag{29}$$

(*****You may want to write this out using $\eta_{\mu\nu}$, and make sure you understand the manipulations here; it may be necessary, for example, to relabel dummy indices *****).

Integrate by parts in the second term, gives

$$\frac{du^\nu}{d\tau} = 0. \tag{30}$$