In a weak gravitational field, we have said:

\[ g_{00}(x) = -(1 + 2\Phi(x)) \quad g_{ij} = (1 - 2\Phi(x))\delta_{ij}. \]  

(1)

Near a point, \( \bar{x}_0 \), we can proceed as follows. First, simply define \( \bar{x}_0 = 0 \), and \( \Phi(x) = gx \) (singling out a direction as the \( x \) direction, for simplicity. The metric is then

\[ ds^2 = dx^2(1 - 2gx) - dt^2(1 + 2gx). \]  

(2)

Consider, then, defining new coordinates, \( x', t' \), by

\[ x = x' - \frac{1}{2}gt'^2 + \frac{1}{2}gx'^2 \quad t = t' - gx't' \]  

(3)

So

\[ dx = dx' + gx'dx' - gt'dt' \quad dt = dt'(1 - gx') - dx'gt'. \]  

(4)

So

\[ ds^2 = dx'^2(1 + 2gx')(1 - 2gx) - dt'^2(1 - 2gx)(1 + 2gx) + dx'dt'(-2g't' + 2gt') + O(t'^2, x'^2, tx). \]  

(5)

Let’s note that this redefinition is

\[ x^\mu = x'^\mu + \frac{1}{2}\Gamma_{\nu\rho}^{\mu}x'^\nu x'^\rho. \]  

(6)

since for this geometry:

\[ \Gamma_{xt}^t \approx g = \Gamma_{tt}^x = -\Gamma_{xx}^x. \]  

(7)

If

\[ ds^2 = dx^2(1 + 2A + 2gx) - dt^2(1 - 2B - 2gx). \]  

(8)

then, for small \( A \) and \( B \), we can make the metric diagonal by the further transformation

\[ x' = x'(1 - A) \quad t = t'(1 + B). \]  

(9)

This is exactly the problem of the Schwarschild geometry, where

\[ ds^2 = -dt^2(1 - \frac{2M}{r}) + (1 - \frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \]

Writing \( r = R + x \), expanding in powers of \( x/R \), we have

\[ ds^2 \approx -(1 - \frac{2M}{R} + \frac{2Mx}{R^2})dt^2 + (1 + \frac{2M}{R} - \frac{2Mx}{R^2})^{-1}dx^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2). \]

If \( M/R \) is small, rescale \( t \rightarrow t + \frac{M}{R}; \quad x \rightarrow x - \frac{M}{R} \), so

\[ ds^2 \approx -(1 + 2gx)dt^2 + (1 - 2gx)dx^2. \]

This is just the metric we encountered above!