Geodesics in space-time

Generalize what we wrote for flat Minkowski space-time

$$S = -m\int d\tau$$

$$\Rightarrow \int \sqrt{g_{\mu\nu}dx^\mu dx^\nu}$$

Motion of particles will extremize S.

Very useful: "inverse metric" $g^{\mu\nu}$

$$g^{\mu\nu} g_{\nu\rho} = \delta^\mu{}_\rho \qquad \left[\text{Ex: } \eta^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\right]$$

For a particle in a weak gravitational field, non-relativistic motion

$$g_{\mu\nu} \simeq \begin{pmatrix} -(1+2\Phi) & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} -(1-2\Phi) & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

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$$S \cong \int dt \sqrt{1-\dot{\vec{x}}^2+2\Phi}$$

$$= \int dt \left(1-\tfrac{1}{2}\dot{\vec{x}}^2+\Phi\right)$$

(multiply by $-m$: action (Newtonian) for a particle in a gravitational field)

The general equation of motion is the geodesic eqn.

$$\ddot{x}^\mu + \Gamma^\mu_{\rho\sigma}\,\dot{x}^\rho\,\dot{x}^\sigma = 0$$

$$\Gamma^\mu_{\rho\sigma} = \frac{1}{2} g^{\mu\alpha}\left\{\frac{\partial g_{\alpha\rho}}{\partial x^\sigma} + \frac{\partial g_{\alpha\sigma}}{\partial x^\rho} - \frac{\partial g_{\rho\sigma}}{\partial x^\alpha}\right]$$

(we will show)

symmetric under $\rho \leftrightarrow \sigma$; $\Gamma^{\overset{4\text{ values}}{\mu}}_{\underbrace{\rho\sigma}_{4\times 5/2 = 10}}$

40 terms

For the weak gravitational field

$$\Gamma^i_{00} = \frac{\partial\Phi}{\partial x^i} \qquad \Gamma^0_{i0} = -\frac{\partial\Phi}{\partial x^i}$$

For non-relativistic motion, $\tau \simeq t$; $u^i \simeq v^i$

$$\frac{d^2 x^i}{dt^2} + \frac{\partial \Phi}{\partial x^i} = 0 \Rightarrow \text{Newton's eqn.}$$

Geodesic eqn. in full generality:

Write

$$S = \int dp \sqrt{-g_{\mu\nu} \frac{dx^\mu}{dp} \frac{dx^\nu}{dp}} \equiv \int dp \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}$$

Call $\quad L = \sqrt{-g_{\mu\nu} \frac{dx^\mu}{dp} \frac{dx^\nu}{dp}} = \frac{d\tau}{dp}$

Euler-Lagrange

$$\frac{d}{dp}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) = \frac{\partial L}{\partial x^\mu}$$

$$\frac{\partial L}{\partial \dot{x}^\mu} = -\frac{g_{\mu\rho} \dot{x}^\rho}{L}$$

$$\frac{\partial L}{\partial x^\mu} = \frac{1}{2} \frac{\partial}{\partial x^\mu} g_{\rho\sigma} \frac{\dot{x}^\rho \dot{x}^\sigma}{L}$$

$\frac{d}{dp} = L \frac{d}{d\tau}$, so

$$L \frac{d}{d\tau}\left(-g_{\mu\rho} \dot{x}^\rho\right) = \frac{1}{2} L \frac{\partial}{\partial x^\mu} g_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma$$

where now $\dot{x}^\rho$, etc., $\equiv \frac{dx^\rho}{d\tau} \quad \left(\frac{1}{L}\frac{dx^\rho}{dp}\right)$

So, using the chain rule

$$g_{\mu\rho}\ddot{x}^{\rho} + \frac{\partial g_{\mu\rho}}{\partial x^{\sigma}}\dot{x}^{\rho}\dot{x}^{\sigma} - \frac{1}{2}\frac{\partial}{\partial x^{\mu}} g_{\rho\sigma}\dot{x}^{\rho}\dot{x}^{\sigma} = 0$$

To get this into a more useful form, multiply by $g^{\nu\mu}$: $\quad (g^{\nu\mu} g_{\mu\rho} = \delta^{\nu}{}_{\rho})$

$$\ddot{x}^{\nu} + \frac{1}{2} g^{\nu\mu}\left(2\frac{\partial g_{\mu\rho}}{\partial x^{\sigma}} - \cancel{\tfrac{1}{2}}\frac{\partial}{\partial x^{\mu}} g_{\rho\sigma}\right)\dot{x}^{\rho}\dot{x}^{\sigma} = 0$$

symmetrize in ρ, σ

$$\ddot{x}^{\mu} + \frac{1}{2} g^{\mu\alpha}\left(\frac{\partial g_{\alpha\rho}}{\partial x^{\sigma}} + \frac{\partial g_{\alpha\sigma}}{\partial x^{\rho}} - \frac{\partial}{\partial x^{\alpha}} g_{\rho\sigma}\right)\dot{x}^{\rho}\dot{x}^{\sigma} = 0$$

(as promised)