The Equations of Macroscopic Electromagnetism

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We have, so far, guessed the equations of macroscopic electrodynamics:

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0. \tag{1}$$

$$\vec{\nabla} \cdot \vec{D} = 4\pi\rho \quad \vec{\nabla} \times \vec{H} - \frac{1}{c}\frac{\partial \vec{D}}{\partial t} = \frac{4\pi}{\mathbf{c}}\vec{J}. \tag{2}$$

We had the relations (“constitutive relations")

$$\vec{D} = \vec{E} + 4\pi\vec{P} \quad \vec{H} = \vec{B} - 4\pi\vec{M}. \tag{3}$$

Goal here is to understand the microscopic origins of these equations. We will use a classical language, but this can readily be translated to quantum mechanics.

Microscopic equations:

$$\vec{\nabla} \cdot \vec{b} = 0 \quad \vec{\nabla} \times \vec{e} + \frac{1}{c}\frac{\partial \vec{b}}{\partial t} = 0. \tag{4}$$

$$\vec{\nabla} \cdot \vec{e} = 4\pi\eta \quad \vec{\nabla} \times \vec{b} - \frac{1}{c}\frac{\partial \vec{e}}{\partial t} = \frac{4\pi}{c}\vec{j}. \tag{5}$$

where for this lecture, the lower case denotes the microscopic quantities.

Some relevant length scales

:

1. Size of nuclei 10^{-13} cm
2. Size of atoms 10^{-8} cm
3. Wavelength of visible light 10^{-6} cm

So for visible light, for example, wave is roughly constant in space over a volume containing a million atoms. For, e.g., x-rays, wavelengths are smaller than typical atomic dimensions, so macroscopic description is inappropriate; x-rays resolve individual atoms.

Averaging

So it makes sense to average over distances large compared to atomic sizes. It is not appropriate to average in time, since, e.g. for light, frequencies are comparable to frequencies of atomic motion.

Introduce a test function, $f(\vec{x})$, and define, for some quantity $F(\vec{x}, t)$ (e.g. one of the fields):

$$\mathcal{F}(\vec{x}, t) \equiv \langle F(\vec{x}, t) \rangle = \int d^3x' f(\vec{x}') F(\vec{x} - \vec{x}', t). \tag{6}$$

Choice of the test function

Would like isotropic, smooth (so, e.g. in momentum space, it doesn't have discontinuities or other irregularities):
Normalize to unity.

$$f(\vec{x}) = (\pi R^2)^{-3/2} e^{-r^2/R^2}. \tag{7}$$

It is clear what this does in coordinate space. What sort of averaging does it do in momentum space?
Good practice with Fourier transforms.

Fourier transform of smoothed quantities

We'd like to see that high wave number (short wavelengths) are smoothed out.

$$\mathcal{F}(\vec{k}, t) = \int d^3x\ e^{-i\vec{k}\cdot\vec{x}} \int d^3x' f(\vec{x}') F(\vec{x} - \vec{x}', t). \tag{8}$$

$$= \int d^3x e^{-i\vec{k}\cdot\vec{x}} \int d^3x' f(\vec{x}') \tag{9}$$

$$\times \int d^3k' \int d^3k'' f(\vec{k}') e^{i\vec{k}'\cdot\vec{x}'} F(\vec{k}'', t) e^{i\vec{k}''\cdot(\vec{x}-\vec{x}')}.$$

Now the $\vec{x}$ and $\vec{x}'$ integrations give δ functions:

$$(2\pi)^6 \delta(\vec{k} - \vec{k}'')\delta(\vec{k}' - \vec{k}''). \tag{10}$$

So the result is

$$f(\vec{k})\mathcal{F}(\vec{k}). \tag{11}$$

It is a useful exercise to calculate the Fourier transform of f. More generally, we will several times be interested in the integral:

$$I(a,\vec{k}) = \int d^3x\ e^{i\vec{k}\cdot\vec{x}} e^{-a^2x^2}. \tag{12}$$

This is a standard integral which is done by completing the squares in the exponent:

$$I(a,\vec{k}) = \int d^3x\ e^{-a^2(\vec{x}-i\frac{\vec{k}}{2a^2})^2-\frac{1}{4}\frac{k^2}{a^2}} \tag{13}$$

$$= \pi^{3/2}a^{-3}e^{-\frac{1}{4}\frac{k^2}{a^2}}.$$

So

$$f(\vec{k}) = \frac{3}{4\sqrt{\pi}} e^{-\frac{1}{4}k^2 R^2}. \tag{14}$$

This means that for wave lengths large compared to R,

$$\mathcal{F}(\vec{k}, t) = F(\vec{k}, t) \tag{15}$$

while for $\lambda \ll R$,

$$\mathcal{F}(\vec{k}, t) \to 0. \tag{16}$$

Note some useful features of this averaging:
Most important, differentiation and averaging commute:

$$\frac{\partial}{\partial x_i}\mathcal{F}(\vec{x},t) = \frac{\partial}{\partial x_i}\int d^3x' f(\vec{x}')F(\vec{x}-\vec{x}',t) \tag{17}$$

$$= \int d^3x' f(\vec{x}')\frac{\partial F}{\partial x_i}(\vec{x}-\vec{x}',t)$$

$$= \langle \frac{\partial F}{\partial x_i} \rangle.$$

Clearly also $\frac{\partial}{\partial t}$ commutes with averaging.

So consider, first, the homogeneous equations. Because differentiation and averaging commute, calling

$$\vec{E} = \langle \vec{e} \rangle \quad \vec{B} = \langle \vec{b} \rangle \tag{18}$$

we have:

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}. \tag{19}$$

For the inhomogeneous equations, we have to be more careful. Consider, first,

$$\vec{\nabla} \cdot \vec{E} = 4\pi \langle \eta(\vec{x}, t) \rangle. \tag{20}$$

We need to convert the right hand side into something recognizable.
First, we divide the charge density into a “free" piece and a bound piece, corresponding to electrons bound in molecules:

$$\eta = \eta_{free} + \eta_{bd} \tag{21}$$

where

$$\eta_{free}(\vec{x}, t) = \sum_{free} q_j \delta(\vec{x} - \vec{x}_j(t)) \quad \eta_{bd}(\vec{x}, t) = \sum_n \eta_n(\vec{x}, t). \tag{22}$$

Here η_n is the microscopic charge density of the n'th molecule:

$$\eta_n(\vec{x}, t) = \sum_{j \in n} q_j \delta(\vec{x} - \vec{x}_j(t)). \tag{23}$$

For the bound part, we first write:

$$\vec{x}_j = \vec{x}_n + \vec{x}_{jn}. \tag{24}$$

Then

$$\langle \eta_n(\vec{x}, t) \rangle = \int d^3x' f(\vec{x}') \eta_n(\vec{x} - \vec{x}', t) \tag{25}$$

$$= \sum q_j \int d^3x' f(\vec{x}') \delta(\vec{x} - \vec{x}' - \vec{x}_{jn} - \vec{x}_n)$$

$$= \sum_{j \in n} q_j f(\vec{x} - \vec{x}_{jn} - \vec{x}_n).$$

Now we see the virtue of a smooth choice of f. For $R \gg$ angstrom, we can Taylor expand f:

$$\langle \eta_n(\vec{x},t)\rangle = \sum_{j\in n} \Big[f(\vec{x}-\vec{x}_n) - \vec{x}_{jn}\cdot\vec{\nabla} f(\vec{x}-\vec{x}_n) \tag{26}$$

$$+\frac{1}{2}\sum_{\alpha\beta}(x_{jn})_\alpha (x_{jn})_\beta \frac{\partial^2}{\partial x_\alpha \partial x_\beta} f(\vec{x}-\vec{x}_n) + \cdots .$$

Now

$$\sum_{jn} q_j = q_n; \quad \sum q_j \vec{x}_{jn} = \vec{p}_n \tag{27}$$

and the last sum in the brackets above is related to the quadrupole moment. Dropping this term as small in most circumstances, we have

$$\langle \eta_n(\vec{x}, t) \rangle = q_n f(\vec{x} - \vec{x}_n) - \vec{p}_n \cdot \vec{\nabla} f(\vec{x} - \vec{x}_n) + \ldots\ldots \quad (28)$$

This is what we would have obtained from

$$\langle \eta_n(\vec{x}, t) \rangle = \langle q_n \delta(\vec{x} - \vec{x}_n) \rangle - \vec{\nabla} \cdot \langle \vec{p}_n \delta(\vec{x} - \vec{x}_n) \rangle + \ldots . \quad (29)$$

(Check!)

So summing over the molecules

$$\langle \eta(\vec{x}, t) \rangle = \rho(\vec{x}, t) - \vec{\nabla} \cdot \vec{P}(\vec{x}, t) + \ldots \tag{30}$$

where

$$\rho(\vec{x}, t) = \langle \sum_{j(free)} q_j \delta(\vec{x} - \vec{x}_j) + \sum_n q_n \delta(\vec{x} - \vec{x}_n) \rangle \tag{31}$$

$$\vec{P}(\vec{x}, t) = \langle \sum \vec{p}_n \delta(\vec{x} - \vec{x}_n) \rangle. \tag{32}$$

So we have

$$\vec{\nabla} \cdot \vec{D}(\vec{x}, t) = \rho_{free}(\vec{x}, t) \tag{33}$$

where

$$\vec{D}(\vec{x}, t) = \vec{E} + 4\pi \vec{P}(\vec{x}, t). \tag{34}$$

It is straightforward to carry out this expansion to higher orders in a/R. This is described in G. Rusasakoff, American Journal of Physics, **38** (1970) 1188 (a rather pretty article). In this case, there are further corrections to $\vec{D}$, e.g.

$$D_\alpha = E_\alpha + 4\pi P_\alpha - 4\pi \sum_\beta \frac{\partial Q'_{\alpha\beta}}{\partial \beta}. \tag{35}$$

Note that there is no assumption here that the system is static!

Now for the last of the Maxwell equations. Here, we need somehow to get $\frac{\partial \vec{D}}{\partial t}$ on the right hand side, with $\vec{D}$ as defined above (including higher order terms! We'll content ourselves with the leading term).

Start, as before, breaking up

$$\vec{j} = \vec{j}_{free} + \vec{j}_{bd}. \quad (36)$$

$$\vec{j}_{bd} = \sum_n \vec{j}_n(\vec{x}, t). \quad (37)$$

Again

$$\vec{j}_n(\vec{x}, t) = \sum_{j \in n} q_j \vec{v}_j \delta(\vec{x} - \vec{x}_j). \quad (38)$$

Again, take

$$\vec{x}_j = \vec{x}_{jn} + \vec{x}_n \quad \vec{v}_j = \vec{v}_{jn} + \vec{v}_j. \tag{39}$$

So

$$\langle j_n(\vec{x}, t)\rangle = \sum_{j\in n} q_j(\vec{v}_{jn} + \vec{v}_n) f(\vec{x} - \vec{x}_n - \vec{x}_{jn}). \tag{40}$$

Again, Taylor expand f about $\vec{x}_n$; we'll stop with the second term:

$$\langle j_n(\vec{x}, t)\rangle = \sum q_j(\vec{v}_{jn}+\vec{v}_n) f(\vec{x}-\vec{x}_n) - \sum q_j(\vec{v}_{jn}+\vec{v}_n)\vec{x}_{jn}\cdot\vec{\nabla} f(\vec{x}-\vec{x}_n) \tag{41}$$

We want to write these quantities in a more familiar form.

$$\sum q_j \vec{v}_{jn} = \frac{d}{dt} \sum q_j \vec{x}_{jn} \tag{42}$$

$$= \frac{d}{dt} \vec{p}_n .$$

Now consider

$$-\sum q_j (v_{jn})_\alpha (x_{jn})_\beta \tag{43}$$

$$= -\frac{1}{2}\sum q_j((v_{jn})_\alpha (x_{jn})_\beta - (v_{jn})_\beta (x_{jn})_\alpha - \text{ symmetric term}$$

The first term can be rewritten in terms of the magnetic moment of the *n*'th molecule, in a way which is now familiar:

$$\vec{m}_n = \frac{1}{2c}\sum_j q_j(\vec{x}_{jn} \times \vec{v}_{jn}). \tag{44}$$

So we have

$$\langle \vec{j}_\alpha \rangle = \langle \vec{j}_{n\alpha} \delta(\vec{x}-\vec{x}_n) \rangle + \langle \frac{d}{dt} \vec{p}_{n\alpha} \delta(\vec{x}-\vec{x}_n) \rangle + c\epsilon_{\alpha\beta\gamma} \partial_\beta \langle \vec{m}_{n\gamma} \delta(\vec{x}-\vec{x}_n) \rangle + \ldots . \quad (45)$$

Defining the macroscopic current density:

$$\vec{J}(\vec{x},t) = \langle \sum_j q_j \vec{v}_j \delta(\vec{x}-\vec{x}_j) \rangle + \langle \sum_n q_n \vec{v}_n \delta(\vec{x}-\vec{x}_n) \rangle \quad (46)$$

and

$$\vec{M}(\vec{x},t) = \langle \sum_n \vec{m}_n \delta(\vec{x}-\vec{x}_n) \rangle \quad (47)$$

$$(\vec{\nabla} \times \vec{B})_\alpha = \frac{4\pi}{c} \vec{J}_\alpha + 4\pi (\vec{\nabla} \times \vec{M})_\alpha + \frac{1}{c} \frac{\partial D_\alpha}{\partial t}. \tag{48}$$

So calling $\vec{H} = \vec{B} - 4\pi \vec{M}$,

$$\vec{\nabla} \times \vec{H} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{D}}{\partial t}. \tag{49}$$