Some aspects of Magnetotistatics

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The complication in the case of magnetism is the vector character of $\vec{A}$ and $\vec{J}$. So the multipole expansion is more complicated to work out. We will describe a general approach which allows a systematic multipole expansion. But we begin by simply expanding our expression for $\vec{A}$ in Cartesian coordinates:

$$\vec{A}(\vec{x}) = \frac{1}{c}\int d^3x' \frac{1}{|\vec{x} - \vec{x}'|}\vec{J}(\vec{x}'). \tag{1}$$

This expression requires some massaging to obtain a transparent expression. Critical is that, for statics, $\vec{\nabla} \cdot \vec{J} = 0$. It is helpful to write this equation with components for the different indices.

$$A_i(\vec{x}) = \frac{1}{c}\int d^3x' \frac{1}{|\vec{x}-\vec{x}'|} J_i(\vec{x}') \tag{2}$$

$$\approx \frac{1}{cr}\int d^3x' (1 + \frac{\vec{x}\cdot\vec{x}'}{r^2}) J_i(\vec{x}').$$

The first term vanishes. This follows by writing

$$\int d^3x' J_i = \int d^3x' \partial'_k (x'_i J_k) = 0 \tag{3}$$

where the first step follows by performing the differentiation, using current conservation, and the last follows from the assumption that the current distribution is localized, plus Gauss's theorem.

The second term may be rewritten in a form which involves the dipole moment. We start by noting that

$$\int d^3x'(x'_i J_j + x'_j J_i) = 0. \quad (4)$$

This follows because the integrand is, in fact, a total derivative; it is equal to

$$\int d^3x' \partial'_k (x'_i x'_j J_k) \quad (5)$$

where, again, we used current conservation.

This allows us to rewrite $\vec{A}$:

$$A_i(\vec{x}) = \frac{1}{cr^3} \int d^3x' \frac{1}{2} x_i (x_i' J_j - x_j' J_i). \tag{6}$$

The last equation has a structure reminiscent of a cross product. Indeed, whenever we have an anti symmetric tensor, F_{ij}, we can make a (pseudo) vector,

$$V_i = \epsilon_{ijk} F_{jk}. \tag{7}$$

We can think of the $\vec{B}$ field this way, where $F_{jk} = \partial_j A_k - \partial_k A_j$.

We can recover F from B, if we like:

$$F_{ij} = \frac{1}{2}\epsilon_{ijk} B_k \tag{8}$$

Check:

$$\frac{1}{2}\epsilon_{ijk}\epsilon_{klm}(\partial_l A_m - \partial_m A_l)$$
$$= \partial_i A_j - \partial_j A_i.$$

In the present case, we define the vector, $\vec{m}$, where

$$\vec{A} = \frac{\vec{m} \times \vec{x}}{r^3} \quad (9)$$

where

$$\vec{m} = \frac{1}{2c} \int d^3x' \times \vec{J}. \quad (10)$$

For the case of a collection of particles of mass m_i, charges q_i, located at $\vec{x}_i$,

$$\vec{J}(\vec{x}) = \sum q_i \vec{v}_i \delta^{(3)}(\vec{x} - \vec{x}_i), \tag{11}$$

we have

$$\begin{aligned} \vec{m} &= \frac{1}{2c} \int d^3x' \vec{x}' \times \vec{J}(\vec{x}') \\ &= \frac{1}{2c} q_i \vec{x}_i \times \vec{v}_i \\ &= \sum \frac{q_i}{2m_i c} \vec{L}_i. \end{aligned} \tag{12}$$

This formula is not quite right for spin; in the case of the electron, for example, the magnetic moment is approximately

$$\vec{m} = \frac{ge}{mc} \vec{S}.$$

Here, the Dirac equation gives $g = 2$, while QED gives corrections in a power series in α. For the muon, g is measured so accurately that one is sensitive to corrections from the strong and weak interactions (it is also calculated as accurately).

Force on a Dipole; Potential of a Dipole

Because there is not an energy associated with a charge in a magnetic field, we examine the force on a localized charge distribution:

$$F_i = \frac{1}{c} \int d^3x \vec{J} \times \vec{B}. \tag{13}$$

Expanding $\vec{B}$ in a Taylor series about the origin:

$$F_i = \frac{1}{c} \epsilon_{ijk} \left[B_k(0) \int J_j(\vec{x}') d^3x' + \int J_j(\vec{x}') \vec{x}' \cdot \vec{\nabla} B_k(0) d^3x' \right]. \tag{14}$$

As before, first integral vanishes; second we rewrite, again, antisymmetrizing in indices:

$$F_i = -\frac{1}{2c}\epsilon_{ijk}\int (x'_\ell J_j - x'_j J_\ell)\nabla_\ell B_k(0) d^3x'. \qquad (15)$$

The expression in partenthesis is $\epsilon_{jkm} m_m$, so using our identities, we have:

$$F = \partial_i(\vec{m}\cdot\vec{B}) \qquad (16)$$

so we can think of a potential:

$$\vec{F} = -\vec{\nabla} U \quad U = -\frac{1}{c}\vec{m}\cdot\vec{B}. \qquad (17)$$

Some aspects of Magnetism in Materials

As for dielectrics, suppose a magnetization density, $\vec{M}(\vec{x})$. Then, allowing also for “free" currents we have

$$\vec{A}(\vec{x}) = \int d^3x' \left[\frac{\vec{j}(\vec{x}')}{|c\vec{x} - \vec{x}'|} + \frac{\vec{M}(\vec{x}') \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} \right] \qquad (18)$$

$$= \int d^3x' \left[\frac{\vec{j}(\vec{x}')}{|c\vec{x} - \vec{x}'|} + \vec{M}(\vec{x}') \times \vec{\nabla}' \frac{1}{|c\vec{x} - \vec{x}'|} \right]$$

$$= \int d^3x' \left[\frac{\vec{J}(\vec{x}')}{|c\vec{x} - \vec{x}'|} + \frac{\vec{\nabla}' \times \vec{M}(\vec{x}')}{|c\vec{x} - \vec{x}'|} \right] .$$

So the magnetization acts as an effective current density,

$$\vec{J}_M = c\vec{\nabla} \times \vec{M}. \tag{19}$$

The equation for curl $\vec{B}$ becomes:

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J} + 4\pi\vec{\nabla} \times \vec{M}. \tag{20}$$

So with $\vec{H} \equiv \vec{B} - 4\pi\vec{M}$,

$$\vec{\nabla} \times \vec{H} = \frac{4\pi}{c}\vec{J} \quad \vec{\nabla} \cdot \vec{B} = 0. \tag{21}$$

To solve these equations, we need a relation between $\vec{B}$ and $\vec{H}$. For most systems, linear,

$$\vec{B} = \mu \vec{H} \tag{22}$$

(the origin, in a sense, of the μ_0 in Jackson, permeability of the vacuum).
$\mu > 1$: paramagnetic
$\mu < 1$: diamagnetic Differences typically part in 10^5 difference from one.
Ferromagnets: much more complicated; hysteresis (depends on history of preparation of system).

Boundary value problems with magnetic materials (*very* brief).

Boundary conditions: if no currents, continuity of tangential $\vec{H}$, normal $\vec{B}$.
$\vec{\nabla} \times \vec{H} = 0$, so

$$\vec{H} = \vec{\Phi}_M \quad \nabla^2 \Phi = 0 \tag{23}$$

and use methods familiar from electrostatics (see Jackson).

Magnetism in Materials (*brief*)

Quantum mechanics required to understand (nice introduction in Feynman, volume 2; see also Eisenberg and Resnick).
Paramagnetism: occurs in systems with unpaired spins. Magnetic moments tend to align with the field ($-\vec{m}\cdot\vec{B}$ potential). Alignment strongest at low temperatures.

$$\vec{M} = \chi\vec{B} \tag{24}$$

$\chi \sim 1/T$ (Curie law).
Diamagnetism: associated with Lenz's law (but also needs quantum mechanics).
Ferromagnetism: not due to magnetic forces, but to forces in atoms which have net effect of aligning the spins. ("Exchange" interactions, in systems with unfilled shells (*d*). Occurs only in iron, cobalt, nickel, gadolinium and dyprosium (and alloys of these materials).