Coupling Charged Particles to the Electromagnetic Field: The Bohm Aharanov Effect, Magnetic Monopoles

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The Lagrangian for Particles in an Electromagnetic Field

We can guess a lagrangian by demanding gauge invariance. Recalling that under gauge transformations:

$$\phi(\vec{x},t) = \phi'(\vec{x},t) - \frac{1}{c}\frac{\partial \Lambda(\vec{x},t)}{\partial t} \quad \vec{A}(\vec{x},t) = \vec{A}'(\vec{x},t) + \vec{\nabla}\Lambda(\vec{x},t) \quad (1)$$

the lagrangian

$$L = \sum_a \frac{m_a}{2}\left(\frac{d\vec{x}_a}{dt}\right)^2 - \quad (2)$$

$$\int d^3x \left(-\phi(\vec{x},t)\rho(\vec{x},t) + \frac{1}{c}\vec{A}(\vec{x},t)\cdot\vec{J}(\vec{x},t)\right)$$

is invariant, due to current conservation.

For a collection of point particles,

$$\rho(\vec{x},t) = \sum_a q_a \delta(\vec{x} - \vec{x}_a(t)) \quad \vec{J}(\vec{x},t) = \sum_a q_a \vec{v}_a \delta(\vec{x} - \vec{x}_a(t)) \quad (3)$$

So, integrating over $\vec{x}$,

$$L = \sum_a \left(\frac{m_a}{2} - q_a \Phi(\vec{x}_a, t) + \frac{q_a}{c} \vec{v}_a \cdot \vec{A}(\vec{x}_a, t) \right) . \quad (4)$$

Note carefully that Φ and $\vec{A}$ are functions of $\vec{x}_a$ and t.

The Lorentz Force Law

If all of this is correct, the Lorentz force law should emerge as the Euler-Lagrange equation for this system. It is convenient, to avoid getting confused about vector quantities, to work with Cartesian indices. We need to evaluate:

$$\frac{\partial L}{\partial x_a^i} = -q_a \left(\vec{\nabla}^i_a \Phi - v_a^j \vec{\nabla}^i_a A^j \right) . \tag{5}$$

After this, we drop the subscript *a* on the partial derivatives, to streamline the writing, i.e.

$$\frac{\partial L}{\partial x_a^i} = -q_a \left(\vec{\partial}^i \Phi - v_a^j \partial^i A^j \right) . \tag{6}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_a^i} = \frac{d}{dt}\left(m_a \dot{x}_a^i + \frac{q_a}{c} A^i\right). \tag{7}$$

To evaluate the time derivative, we need to keep in mind that this is the total derivative, i.e.

$$\frac{dA_i}{dt} = \frac{\partial A_i}{\partial t} + v_j \partial_j A^i. \tag{8}$$

So, now dropping the *a* subscript everywhere, which no longer can lead to any ambiguity:

$$m\ddot{x}^i = q(-\partial_i \Phi - \frac{1}{c}\partial_t A^i) + \frac{q}{c}(+v^j \partial_i A^j - v^j \partial_j A^i). \tag{9}$$

The first term we recognize immediately as giving us $q\vec{E}$; the second is readily seen (using our favorite identity) to be $q\vec{v} \times \vec{B}$. So indeed this lagrangian gives us the Lorentz force for each charged particle!

The Hamiltonian: passing to quantum mechanics

If we are interested in quantum mechanics, we need to construct the Hamiltonian. For this, we first need the *canonical momentum*.

$$p_a^i = \frac{\partial L}{\partial x_a^i} = m\dot{x}_a^i + \frac{1}{c}A^i(\vec{x}_a, t). \tag{10}$$

Now we compute:

$$H = \sum q_a^i p_a^i - L = m\dot{x}_a^{i\,2} + q_a\phi(\vec{x}_a, t). \tag{11}$$

(We have written the argument of ϕ explicitly, again).

Not surprisingly, $\vec{A}$ does not appear in this expression. But the Hamiltonian is not to be thought of as a function of the velocities, but of the coordinates and momenta. So

$$H = \sum \left(\frac{1}{2m_a} (\vec{p}_a - q_a \vec{A})^2 + q_a \Phi \right) \quad (12)$$

In passing to quantum mechanics, one takes

$$\vec{p}_a \to -i\hbar \vec{\nabla}_a \quad (13)$$

so $\vec{A}$ appears in the Schrodinger equation.

Gauge invariance of the Schrodinger Equation

What happened to gauge invariance?

Consider

$$H\Psi = i\frac{\partial}{\partial t}\Psi. \tag{14}$$

The full gauge invariance is now:

$$\phi(\vec{x},t) = \phi'(\vec{x},t) - \frac{1}{c}\frac{\partial\Lambda(\vec{x},t)}{\partial t} \quad \vec{A}(\vec{x},t) = \vec{A}'(\vec{x},t) + \vec{\nabla}\Lambda(\vec{x},t) \tag{15}$$

$$\Psi = e^{i\frac{q\Lambda}{\hbar}}\Psi'.$$

as you can readily check.

The Aharanov-Bohm Effect

Consider an infinite solenoid. Outside, there is no $\vec{B}$ field, but inside, there is a net flux of $\vec{B}$. As a result, the line integral of $\vec{A}$ around the loop is non-zero:

$$\oint d\vec{\ell} \cdot \vec{A} = \oint d\phi A_\phi \sin\theta r = \Phi_M. \tag{16}$$

Despite the fact that $\vec{B}$ vanishes outside, this is a gauge-invariant statement. This follows from the fact that, under a gauge transformation, the left hand side is

$$\oint d\vec{\ell} \cdot \vec{A} = \oint \vec{\nabla}\Lambda \cdot d\vec{\ell} = \Lambda(\phi + 2\pi) - \Lambda(\phi). \tag{17}$$

So as long as the gauge transformation is periodic (single-valued), the integral is gauge invariant.

Since Φ_M is non-zero, on the other hand, $\vec{A}$ must be a gauge transformation of $\vec{A} = 0$; in this way, the $\vec{B}$ field vanishes. Λ is not periodic, so it is not *really* a gauge transformation:

$$\vec{A} = \vec{\nabla}\tilde{\Lambda}; \quad \tilde{\Lambda} = \Phi_M \phi. \tag{18}$$

$\tilde{\Lambda}$ is not a gauge transformation because it is not periodic.

You are have some familiar with the Feynman path integral. We want to consider the effect on the interference pattern, say in a double slit experiment, of the solenoid. In general, Feynman tells us that the amplitude to go from a point A to a point B along a particular classical path is $e^{iS(A,B)/\hbar}$. We have seen how the action is modified in the presence of a $\vec{A}$ field. So if we turn on the solenoid, we the amplitude changes to

$$e^{iS(A,B)/\hbar} e^{i\frac{q}{\hbar}\int_A^B d\vec{\ell}\cdot\vec{A}}. \tag{19}$$

So if we compare to paths, one of which goes around one side of the solenoid, and the other around the other, ending up at the same point, they differ in phase (from their previous difference in phase) by

$$e^{i\oint d\vec{\ell}\cdot\vec{A}} = e^{i\Phi_M/\hbar}. \tag{20}$$

When squaring the sum of these amplitudes, there are new (observable) interference effects.This is the Aharanov-Bohm effect.

In this light, one can understand the Dirac quantization condition for electric charge. We have seen that if monopoles exist, they are described by singular field configurations. This singularity is seemingly a gauge artifact. It can be chosen, for example, to lie in different directions by making a gauge transformation. But this gauge transformation will have observable consequences if it leads to modified interference patters for particles of charge q. The string is detectable, like the Bohm-Aharanov solenoid, in general. More precisely, the singularity can be written:

$$\vec{A} = \vec{\nabla}\Lambda \quad \Lambda = 2g\phi. \tag{21}$$

$$e^{\frac{iq}{\hbar}\oint d\vec{\ell}\cdot\vec{\nabla}\Lambda} = 1. \tag{22}$$

This means:

$$\frac{2gq}{\hbar} = n. \tag{23}$$

This (with variants) is the only known explanation for the quantization of electric charge.