Energy Loss by Charge Particles Passing Through Matter

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Simplified Derivation of Energy Loss Formula

One of the most important problems in experimental physics is the problem of understanding how particles slow down as they pass through matter. Consider a massive charged particle (e.g. a proton or muon) passing through matter. As it passes an electron, it will be minimally deflected (little momentum transfer) but it will give significant amounts of energy to the electron. To start, we will assume that the electron is initially at rest (essentially the assumption that the collision takes place in a small time compared to an atomic period; we can check afterwards if the momentum transferred to the electron is large compared to that of a typical bound electron).

Consider from the point of view of the frame of the electron. Suppose that the heavy particle (we will call it a muon) moves with velocity *v* in the *x* direction; it passes with impact parameter $\vec{b} = b\hat{y}$. Then in the electron frame, the electric fields are (*x* is the direction of the muons motion; recall we obtained this formula earlier by boosting the Coulomb field):

$$E_1 = \frac{e\gamma vt}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \tag{1}$$

The net change of the electron momentum in the direction is:

$$\delta p_2 = \int_{-\infty}^{\infty} dt e E_2(t) dt. \qquad (2)$$

$$= e^2 \gamma b \int_{-\infty}^{\infty} \frac{dt}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}$$

$$= \frac{2e^2}{bv}.$$

So, as a function of the impact parameter, we have:

$$\Delta E(b) = \frac{(\delta p)^2}{2m} = \frac{2e^4}{mv^2 b^2}. \qquad (3)$$

This formula, however, is only valid for a limited range of impact parameters. There is an upper limit on b, $b_{\max}$, coming from the requirement that the time over which the field is appreciable be small compared to $1/\omega_0$, the characteristic time of the atomic motion. To be precise, in the language of quantum mechanics, if the time is much longer than this, the field changes adiabatically from the viewpoint of the bound electron, and no energy is transferred. This characteristic time, we see from the form of the field, is

$$t_c \sim \frac{b}{\gamma v} \tag{4}$$

so

$$b_{\max} = \frac{\gamma v}{\omega_0}. \tag{5}$$

There is also a minimum *b* for which this treatment breaks down, because we don't correctly estimate the momentum transfer with our non-relativistic formula. To estimate this, consider a head on collision. From the point of view of the heavy particle rest frame, the electron scatters back, and we can neglect the heavy particle recoil. The energy transferred to the electron (in the electron frame – there is essentially no energy transfer in the massive particle's rest frame) is:

$$\Delta E = \gamma\beta(p_1 - p_2) \tag{6}$$
$$\approx 2m\gamma^2\beta^2$$

This, then, gives a cutoff on *b*, when

$$\frac{2e^4}{mb^2v^2} = 2m\gamma^2v^2 \tag{7}$$

or

$$b_{min} = \frac{e^2}{\gamma mv^2}. \tag{8}$$

(Note: the picture of the electron bouncing off the muon, with essentially no muon recall, is valid provide the center of mass energy is small compared to the muon rest mass; the center of mass energy is of order $\sqrt{mE_\mu}$, which in the case of the muon is small provided the $E_\mu < 100m_\mu$; the requirement is even more stringent for protons).

From this we can compute the energy loss of the particle per unit length:

$$\frac{dE}{dx} = 2\pi NZ \int_{b_{min}}^{b_{max}} \Delta E(b) b\, db \tag{9}$$

$$\approx 4\pi NZ \frac{e^4}{mv^2} \ln(\gamma^2 mv^3 / e^2 \omega_0).$$

Note that this expression is sensitive to the *energy* of the particle (γm). So energy loss is often used to give an energy measurement (momentum being obtained from the curvatures of tracks in a magnetic field, for example).

This formula has a number of limitations, however, and requires significant modifications. Important effects include:

1. quantum mechanics: discrete energy transfers
2. binding effects
3. effects due to coherent scattering off atoms (density effect).

We can make a simple classical model of bound state effects which incorporates correctly some of the features of the real problem by considering our usual favorite, the harmonically bound charge. Here we have:

$$\vec{x}(\omega) = -\frac{e}{m}\frac{\vec{E}(\omega)}{\omega_0^2 - \omega^2 - i\omega\Gamma} \tag{10}$$

so we can compute the energy transfer:

$$\Delta E = \int dt \frac{dE}{dt} = \int dt \vec{E} \cdot \vec{J} d^3 x' \tag{11}$$

$$= -\int dt e \vec{v} \cdot \vec{E}.$$

In terms of the Fourier representations:

$$(\vec{x}(t), \vec{E}(t)) = \int \frac{d\omega}{2\pi} e^{-i\omega t}(\vec{x}(\omega), \vec{E}(\omega)) \qquad (12)$$

$$\Delta E = 2e\mathrm{Re} \int_0^\infty (i\omega\vec{x}(\omega \cdot \vec{E}^*(\omega)d\omega. \qquad (13)$$

Substituting our expression for $\vec{x}$, gives an expression which can be integrated in the limit of small Γ (little dissipation; in the quantum mechanical context, this means long lifetime or narrow width):

$$\Delta E = \frac{e^2}{m} \int_0^\infty |E(\omega)|^2 \frac{2\omega^2\Gamma}{[(\omega_0^2 - \omega^2)^2 + \omega^2\Gamma^2]} d\omega \qquad (14)$$

$$\approx \frac{e^2}{m} |E(\omega_0)|^2 \int_0^\infty \frac{x^2 dx}{(x^2 - \omega_0^2/\Gamma^2)^2 + x^2}$$

$$= \frac{\pi e^2}{m} |E(\omega_0)|^2.$$

Exercise: Verify the equations above.

Now one just needs to compute the Fourier transform:

$$E_2(\omega) = eb\gamma \int \frac{e^{i\omega t}dt}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}. \tag{15}$$

Calling $x = \gamma vt/b$, the integral can be rewritten:

$$e \int_{-\infty}^{\infty} \frac{e^{i\omega bx/\gamma v}}{(1 + x^2)^{3/2}}dx = e \int_{-\infty}^{\infty} \frac{e^{i\xi x}}{(1 + x^2)^{3/2}} \tag{16}$$

where

$$\xi = \omega b/\gamma v = b/b_{max}. \tag{17}$$

When $\xi \ll 1$, this is exactly the integral we encountered previously. When your text encounters integrals like this, it rewrites them in terms of modified Bessel functions.

For large ξ, the integral can be estimated. Note that the integrand has branch cuts at $x = \pm i$. One can view this as an integral in the complex plane, and deform so it encircles the branch cuts. This permits a demonstration that the integral dies exponentially as $e^{-\xi}$.

In any case, a more complete analysis including quantum mechanical effects, due originally to Bethe (the simple classical analysis is due to Bohr) yields an expression:

$$\frac{dE}{dx} = 4\pi N Z z^2 \frac{e^4}{mv^2} \left[\ln(\frac{z\gamma^2 mv^2}{\hbar\omega}) - v^2/c^2 \right] \tag{18}$$

where ze is the charge of the heavy particle (it is easy to redo our formulas allowing for a particle of different charge, such as an α particle or a hypothetical particle of fractional charge (I once was interested in particles of charge $1/7$). You should study figure in Jackson; note initial decrease with β, then growth and plateau.

Another Derivation, Following Landau and Lifschitz

Landau and Lifschitz (in *Electrodynamics of Continuous Media*, on reserve) suggest a different strategy. Consider the force on the moving particle due to its own field. To make sense of this notion, subtract off the corresponding expression in vacuum ($\epsilon = 1$).

We'll just do this for the non-relativistic case. With

$$\rho(\vec{x}, t) = e\delta(\vec{r} - \vec{v}t) \tag{19}$$

we can solve for the Fourier transform (in space) of ϕ. We have, first

$$\rho(\vec{k}, t) = e \int d^3r \delta(\vec{r} - \vec{v}t)e^{-i\vec{k}\cdot\vec{r}} \tag{20}$$

$$= ee^{-i\vec{k}\cdot\vec{v}t}.$$

So

$$\phi(\vec{k}, t) = \frac{4\pi e}{k^2\epsilon(\vec{k} \cdot \vec{v})}e^{-it\vec{v}\cdot\vec{k}}. \tag{21}$$

In the non-relativistic limit, we have:

$$\vec{E}(\vec{k}) = -\frac{4\pi i e \vec{k}}{k^2 \epsilon (\vec{k} \cdot \vec{v})} e^{-it\vec{v}\cdot\vec{k}} \tag{22}$$

and

$$\vec{E}(\vec{r}) = \int \vec{E}(\vec{k}) e^{i\vec{k}\cdot\vec{r}} \frac{d^3k}{(2\pi)^3}. \tag{23}$$

Now the exponential here (remember $\vec{r} = \vec{v}t$ is the position of the particle) cancels the exponential in $\vec{E}(\vec{k})$, and we are left with:

$$\vec{F} = -4\pi i e^2 \int \frac{\vec{k}d^3k}{(2\pi)^3 k^2 \epsilon(\vec{k} \cdot \vec{v})} \tag{24}$$

Now change variables. Call $k_x v = \omega$, $q = \sqrt{k_y^2 + k_z^2}$. Evaluate $F = F_x$:

$$F = \frac{ie^2}{\pi} \int_{-\infty}^{\infty} d\omega \int_0^{q_0} \frac{q\omega dq d\omega}{\epsilon(\omega)(q^2 v^2 + \omega^2)}. \tag{25}$$

q_0 we expect to be of order $1/b_{min}$. The q integral is elementary, giving:

$$F = \frac{ie^2}{\pi v^2} \int_{-\infty}^{\infty} \omega \frac{1}{\epsilon(\omega)} \ln(q_0 v/\omega) d\omega. \tag{26}$$

Now we take my favorite model for $\epsilon(\omega)$:

$$\epsilon(\omega) = 1 + \frac{Ne^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} \tag{27}$$

so

$$\frac{1}{\epsilon} - 1 = \frac{Ne^2}{m(\omega_0^2 - \omega^2 - i\omega\gamma)} \times \frac{1}{1 + \frac{Ne^2}{m(\omega_0^2 - \omega^2 - i\omega\gamma)}}. \tag{28}$$

So F can be studied as a contour integral (in the second factor of the expresson. The integrand (i.e. the numerator of the equation above) has poles just above and just below the real axis, for small γ.

You should check these are at:

$$\omega \approx \omega_0 - \frac{i\gamma}{2} \quad \omega = -\omega_0 + i\frac{\gamma}{2}. \tag{29}$$

Doing the contour integral, and closing above, gives

$$F = \frac{e^2}{\pi v^2} N e^4/m \ln(v^3/\omega_0 e^2) \tag{30}$$

in agreement with our earlier results. While this may appear to be sleight of hand, it has similar assumptions as in the earlier analysis. In particular, the cutoff b_{max} arose from requiring that the disturbance *not* be adiabatic. In quantum mechanics, if the process is not adiabatic, there are transitions between states, and one expects the width (γ) to be important. Note that it actually cancels out of the final answer, but plays a role in the intermediate stages of the computation.

Exercise: Starting from the Poisson equation in momentum space, verify the sequence of equations above leading to equation 30. You should feel free to critique the derivation.

Density Effect and Cherenkov Radiation

A further modification is important at high densities. Here the issue is that the particle scatters off of several electrons at once. The atom responds to the field of the passing particle, modifying the field. This is best described in the language of dielectrics. The derivation below has much in common with the derivation of the basic energy loss formula of the previous section.

In Fourier space (I'll adopt Jackson's convention for the Fourier transformation) we have the equations:

$$[k^2 - \frac{\omega^2}{c^2}\epsilon(\omega)]\Phi(\vec{k}, \omega) = \frac{4\pi}{\epsilon(\omega)}\rho(\vec{k}, \omega) \tag{31}$$

$$[k^2 - \frac{\omega^2}{c^2}\epsilon(\omega)]\vec{A}(\vec{k}, \omega) = 4\pi\vec{J}(\vec{k}, \omega) \tag{32}$$

Fourier transforming ρ and $\vec{J}$ for the charged particle: (We follow Jackson's convention for Fourier transforms,

$$F(\vec{x}, t) = \frac{1}{(2\pi)^2} \int d^3k \int d\omega F(\vec{k}, \omega) e^{i\vec{k}\cdot\vec{k} - i\omega t} \tag{33}$$

which is not my preference). Then

$$\rho(\vec{x}, t) = e\delta(\vec{x} - \vec{v}t) \quad \vec{J}(\vec{x}, t) = \vec{v}\rho(\vec{x}, t). \tag{34}$$

$$\rho(\vec{k}, \omega) = \frac{e}{2\pi}\delta(\omega - \vec{k} \cdot \vec{v}) \quad \vec{J}(\vec{k}, \omega) = \vec{v}\rho(\vec{k}, \omega). \tag{35}$$

So we can immediately write the potentials (in Fourier space, the problem is algebraic):

$$\vec{\Phi}(\vec{k}, \omega) = \frac{2e}{\epsilon(\omega)} \frac{\delta(\omega - \vec{k} \cdot \vec{v})}{k^2 - \frac{\omega^2}{c^2}\epsilon(\omega)} \tag{36}$$

and

$$\vec{A}(\vec{k}, \omega) = \epsilon(\omega)\frac{\vec{v}}{c}\Phi(\vec{k}, \omega). \tag{37}$$

The electric and magnetic fields are relatively simple in momentum space:

$$\vec{E}(\vec{k}, \omega) = i\left[\frac{\omega\epsilon(\omega)}{c} - \vec{k}\right]\Phi(\vec{k}, \omega) \tag{38}$$

and

$$\vec{B}(\vec{k}, \omega) = i\epsilon(\omega)\vec{k} \times \frac{\vec{v}}{c}\Phi(\vec{k}, \omega) \tag{39}$$

To determine the energy loss:

$$\Delta E = -e \int_{-\infty}^{\infty} \vec{v} \cdot \vec{E} dt = 2e \mathrm{Re} \int_0^{\infty} i\omega \vec{x}(\omega) \cdot \vec{E}^*(\omega) d\omega. \quad (40)$$

$\vec{E}$ is the Fourier transform of $\vec{E}$ with respect to time at the location $\vec{x}$; this is a distance b from the trajectory of the charged projectile, i.e.

$$\vec{E}(\omega) = \frac{1}{(2\pi)^{3/2}} \int d^3k \vec{E}(\vec{k}, \omega) e^{ibk_3}. \quad (41)$$

This integral can be expressed in terms of modified Bessel functions, but the asymptotic behavior for large b can be studied using the contour integral methods we used above. The final result is:

$$\frac{dE}{dx} = \frac{(e)^2 \omega_p^2}{c^2} \ln(1.123c/a\omega_p). \tag{42}$$

Here $\omega_p^2 = \frac{4\pi NZe^2}{m}$. (See p. 636 of Jackson for a graphical description; the density effect tends to decrease the energy loss at low density).

Cherenkov Radiation

A very interesting effect associated with the non-trivial index of refraction. A particle moving at faster than the speed of light in a medium emits radiation (even without acceleration) in a cone of a characteristic size.

The cone is easy to understand from the formulas we have written above. From our expressions for Φ and $\vec{A}$, we see that

$$k_z v = \omega \tag{43}$$

this is

$$k v \cos \theta = k/n \tag{44}$$

or

$$\cos \theta = c/nv \tag{45}$$

($n = n(\omega)$ is a function of frequency).

To actually verify that there is emission requires using our previous formulas for the fields and the energy loss. The result can be written:

$$\frac{dE}{d\omega} = \frac{e^2}{c^2} \left(1 - \frac{c^2}{v^2 n^2} \right) \omega d\omega. \tag{46}$$

Note, in particular, that the rate grows with ω. This is related to the blue light observed in nuclear reactor cooling water. Cherenkov radiation, because of the sensitivity of the cone to the velocity, is an important particle detection technique.

We can derive the energy loss by a technique similar to that we used previously. Using our expression for the field, E_x, as the Fourier integral of $\vec{E}(\vec{k})$, we can write

$$F = eE_z = \frac{ie^2}{\pi} \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{\left(1/v^2 - \epsilon/c^2\right) \omega q \, dq \, d\omega}{\left[q^2 + \omega^2 \left(1/v^2 - \epsilon/c^2\right)\right]} \quad (47)$$

where we have made the same substitutions as before. Again, we write as a contour integral. We'll actually do the contour integral as an integral over q.

Write

$$F = \sum \frac{ie^2}{\pi}\epsilon \int_0^\infty d\omega\omega \int_0^\infty \frac{\left(1/v^2 - \epsilon/c^2\right)\omega q dq d\omega}{\left[q^2 + \omega^2\left(1/v^2 - \epsilon/c^2\right)\right]} \qquad (48)$$

where the sum runs over $\omega = \pm|\omega|$ (note the integration limits on ω). Call the expression in the denominator, ξ:

$$\xi = q^2 + \omega^2\left(1/v^2 - \epsilon/c^2\right) \qquad (49)$$

Now use the fact that ϵ is complex (see our simple model), with a small imaginary part, which is positive for $\omega > 0$, negative for $\omega < 0$. So the sum of the two integrals, $\int \frac{d\xi}{\xi}$, yields a line passing above and below the pole at the origin, so giving $2\pi i$. This immediately gives the expression above for $dF/d\omega$.

Exercise: Verify the contour integral manipulations above.