Physics 214. Electricity and Magnetism Professor Dine

Winter 2010: FINAL EXAM

Do 5 of 6 Problems. Sixth will be graded for extra credit.

Problem 1
Suppose $\vec{x}(t)$, $\vec{E}(t)$ have Fourier transforms:

$$\vec{x}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} \vec{x}(\omega) d\omega \qquad \vec{E}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} \vec{E}(\omega) d\omega \tag{1}$$

a. What does the reality of $\vec{x}$ and $\vec{E}$ say about $\vec{x}(\omega)$, $\vec{E}(\omega)$.

Solution: Just use $\vec{x} = \vec{x}^*$, and substitute the Fourier transform on each side. $\vec{x}(\omega) = -\vec{x}^*(-\omega)$, similarly for $\vec{E}$.
b. Write an expression for

$$\int \vec{x}(t) \cdot \vec{E}(t) dt \tag{2}$$

as an integral over ω.

Solution:

Simply plug in the Fourier transforms for each field, being careful to remember that one needs a separate (dummy) integration variable for $\vec{x}$ and $\vec{E}$:

$$\frac{1}{2\pi} \int dt d\omega d\omega' e^{i(\omega+\omega')t} \vec{x}(\omega) \cdot \vec{E}(\omega') \tag{3}$$

The time integral is $\delta(\omega + \omega')$, so we obtain:

$$\int d\omega \vec{x}(\omega) \cdot \vec{E}^*(\omega). \tag{4}$$

(Note that with the convention for 2π's here, there are no 2π's in this expression).

Problem 2

The Stress Tensor

a. Compute the 00, $0i$ and ij components of the stress tensor:

$$T^{\mu\nu} = \frac{1}{4\pi}(F^{\mu\lambda} F_\lambda{}^\nu + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}) \tag{5}$$

Solution: These can be found in Jackson, p. 609. The main thing to be careful about is that $F_{ij}F^{ij} = 2\vec{B}^2$. On has (12.114):

$$T^{00} = \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2) \qquad T^{0i} = \frac{1}{4\pi}(\vec{E} \times \vec{B})_i \qquad T^{ij} = -\frac{1}{4\pi}\left[E_i E_j + B_i B_j - \frac{1}{2}\delta_{ij}(E^2 + B^2)\right] \tag{6}$$

b. Show that the $\nu = i$ component of the equation

$$\partial_\mu T^{\mu\nu} = 0 \tag{7}$$

expresses the conservation of the momentum of the electromagnetic field in free space. (You can use your knowledge of the Poynting vector and the energy density of the electromagnetic field from last quarter).

Solution: Here just integrate the equation of a volume, V, and use Gauss's theorem to write, taking the $\nu = i$ component of the equation as an example:

$$\frac{d}{dt}\int_V d^3xT^{0i} = -\int_S d^2an_jT^{ij} \tag{8}$$

so we have a conservation law for each component of $P^\mu = \int d^3xT^{0\mu}$.
c. Evaluate the components of the momentum flux and T_{ij} for a linearly polarized wave moving along the z axis. Discuss.

Solution: Here the main point is that one doesn't expect momentum flow in *any* of the directions, for a plane wave. So consider, e.g., a wave moving along the z axis and with electric field along the x axis. The B field then has the same magnitude, and is oriented along the y axis. It is easy to see that in fact T^{ij} vanishes in all of the directions.

Problem 3

Relativistic kinematics:

a. The center of mass energy of the LHC, when it is finally working properly, should be 14 TeV (14,000 GeV). What would be the energy required of a fixed target machine (one proton of momentum p incident on another at rest) in order to have the same center of mass energy. You can approximate the rest energy of the proton as 1 GeV.

Solution: Here we use that:

$$E_{cm}^2 = (p_1 + p_2)^2 = 2m^2 + 2p_1 \cdot p_2 \tag{9}$$

$$\approx 2p \times m$$

so

$$p \approx 10^8 \text{ GeV} \tag{10}$$

b. Cosmic rays of sufficiently high energy scattering off the cosmic microwave background photons (energy 10^{-4} eV), can produce pi mesons (energy 130 MeV). These rays cannot reach the other. Consider the case of cosmic ray photons. What is the energy cutoff (known as the GKZ cutoff).

Solution: Now we want:

$$m_\pi^2 = (p_{cr} + p_{cmb})^2 = 2p_{cr} \cdot p_{cmb} \approx 4 \times 10^{-13}p. \tag{11}$$

So

$$p \approx 10^{10} \text{ GeV}. \tag{12}$$

To get some sense what the data shows, see the particle data group review of cosmic rays: http://pdg.lbl.gov/2009/reviews/rpp2009-rev-cosmic-rays.pdf. You can find plots showing evidence for the GZK cutoff. A few years ago, there was datsa from one experiment, as explained there, which suggested that there were significant numbers of events above the cutoff. This was soon contradicted by another experiment, and by continuing work on the first (the issue was probably associated with questions of measuring the energy; because of the very rapid fall off of the cosmic ray flux, small errors in the energy measurement translate into appreciable differences in the number of events).

Problem 4

Radiation

A scalar field obeys the wave equation with a source:

$$\left(\vec{\nabla}^2 - \frac{\partial^2}{\partial t^2}\right)\phi(\vec{x},t) = \rho(\vec{x},t). \tag{13}$$

a. Suppose that ρ is localized in space. Develop a moment expansion for ϕ, at fixed frequency, using the Green's function for the Helmholtz equation (below). Discuss the behavior in the "intermediate zone" and the "radiation zone".

Solution: The scalar field provides a much simpler model for the multipole radiation problem discussed in chapter 9. It has the same features, without so many indices. Here one proceeds just as in chapter 4, writing the solution of the Helmholtz equation, now, instead of the Laplace equation. As there, in both of the regions which concern us, $r_<$ is $|\vec{x}'|$, and $r_> = r$. In the intermediate zone, the result is *exactly* as in the multipole expansion. In particular, using the short distance expansions of j_ℓ and h_ℓ in both regimes, one has precisely the same formulas (the $(2\ell-1)!!/(2\ell+1)!! = \frac{1}{2\ell+1}$ just gives exactly the factor in the static definitions, except that there is now a factor of $1/k^{-\ell}$). At large distances, one gets the moments times factors of $e^{\frac{ikr}{r}}$ and relative factors of $k^{-\ell}$. So the large distance expansion is in powers of d/λ.
b. Suppose that the energy flux is

$$\frac{1}{4\pi}(\partial_i\phi\partial_0\phi). \tag{14}$$

Suppose that the lowest order $\ell = 0$ moment of ρ vanishes. Compute the energy radiated from the $\ell = 1, m = 0$ moment.

Solution: The analog of the pointing vector is

$$\frac{1}{8\pi}(\partial_i\phi^*\omega\phi) \tag{15}$$

We just need the fact that the $\ell = 1, m = 0$ mode is like polarization along the z axis; $\phi \propto \cos\theta\frac{e^{ikr}}{r}$. The derivative along the z axis gives, writing $\cos(\theta) = z/r$, and noting that we better take the derivative to act on the exponential factor, an additional factor of $\hat{r}_i$, so when we dot with $r^2\hat{z}$, we are left with $\cos^2\theta$, which can readily be integrated over $d\Omega$.
Problem 5

Lienard-Wiechart Potentials:

a. Derive the Lienard-Weichart expression for $A^0 = \phi$,

$$\phi(\vec{x},t) = \int d^3x'dt'\frac{1}{|\vec{x}-\vec{x}'|}\delta(t-t'-|\vec{x}-\vec{x}'|)\delta(\vec{x}-\vec{x}_0(t')) \tag{16}$$

Solution: This is in the handout.
b. In our discussion of these potentials and the fields which arise from them we needed

$$\partial_i t_{ret} = \hat{n}_i(1-\hat{n}\cdot\vec{v})^{-1} \tag{17}$$

where $\hat{n} = \frac{\vec{\mathcal{R}}}{\mathcal{R}}$. Derive this.

Solution: This is in the handout.
c. Consider a particle moving along the x axis with velocity v. Take, as in Jackson, the observation point at $x = 0, y = 0, z = b$ at time t. Determine t_{ret} as a function of t and b, and verify equation 17 above.

Solution: The solution for t_{ret} is given in the handout:

It is straightforward algebra to differentiate with respect to b (the derivative with respect to other coordinates vanishes, by symmetry) and verify the formula. The manipulations are again as in the handout. From

$$\mathcal{R} - \vec{v} \cdot \vec{R} = \gamma^{-2}\sqrt{b^2\gamma^{-2} + v^2t^2} \tag{18}$$

and $t_{ret} = \gamma^2 t - \sqrt{b^2\gamma^{-2} + v^2t^2}$. So

$$\frac{\partial t_{ret}}{\partial y} = \frac{\partial t_{ret}}{\partial b} = -\frac{b\gamma^{-2}}{\sqrt{b^2\gamma^{-2} + v^2t^2}} = \frac{R_i}{\mathcal{R} - \vec{v} \cdot \vec{R}}. \tag{19}$$

Problem 6

Energy Loss in Materials:

Consider a scalar field, which obeys an equation similar to the electric and magnetic fields in a medium. In particular, for a slowly moving particle, suppose

$$\nabla^2\phi = -4\pi\rho/\epsilon(\omega). \tag{20}$$

a. For $\rho(\vec{x}, t) = g\delta(\vec{x} - \vec{v}t)$ compute $\rho(\vec{k}, \omega)$.

Solution: From the definition of the Fourier transform,

$$\rho(\vec{k}, \omega) = \frac{1}{(2\pi)^2}\int d^3x dt e^{i\omega t} e^{-i\vec{k}\cdot\vec{x}} g\delta(\vec{x} - \vec{v}t) \tag{21}$$

$$= \frac{1}{(2\pi)^2}\int dt e^{i\omega t - i\vec{k}\cdot\vec{v}t}$$

In lecture, at this stage, we simply identified ω with $\vec{x} \cdot \vec{v}$, but here we can do the integral over t, giving

$$\frac{1}{(2\pi)}\delta(\omega - \vec{v} \cdot \vec{k}). \tag{22}$$

b. Solve for ϕ as a function of $\vec{k}, \omega$.

Solution: This part is easy. In momentum space we have:

$$(-k^2\phi) = g\delta \tag{23}$$

Useful formulae

1. Green's function for the Helmholtz equation:

$$G(\vec{x}, \vec{x}') = ik\sum_{\ell,m} j_\ell(r_<)h_\ell^{(1)}(r_>)Y_{\ell m}(\Omega')Y^*_{\ell m}(\Omega).$$

   At short distances, $j_\ell(x) = \frac{x^\ell}{(2\ell+1)!!}; h_\ell^{(1)} = i\frac{(2\ell-1)!!}{x^{\ell+1}}$, while at large distances, $h_\ell^{(1)} = (-i)^{\ell+1}\frac{e^{ix}}{x}$.

2. $\frac{1}{1+\epsilon} \approx 1 - \epsilon$.

3. Lorentz transformation: $x' = \gamma(x + vt) \quad t' = \gamma(t + vx)$

4. Maxwell's Equations in vacuum (Rationalized MKS):

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_o} \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{B} = \mu_o \vec{J} + \mu_o \epsilon_o \frac{\partial \vec{E}}{\partial t}$$

5. Maxwell's Equations in vacuum (Gaussian):

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{B} = 4\pi \vec{J} + \frac{\partial \vec{E}}{\partial t}$$

6.

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}).$$

7.

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{V}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{V}) - \vec{\nabla}(\vec{\nabla}^2 \vec{V}).$$

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

8. Representation of Delta Function:

$$\delta(t) = \frac{1}{2\pi} \int d\omega e^{-i\omega t}$$

9. A convention for Fourier transforms:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int d\omega e^{-i\omega t} f(\omega)$$

$$f(\omega) = \frac{1}{\sqrt{2\pi}} \int dt e^{i\omega t} f(\omega)$$

10. Energy density of the electromagnetic field:

$$u = \frac{c}{4\pi}(\vec{E}^2 + \vec{B}^2)$$

11. Poynting vector:

$$\vec{S} = \frac{c}{8\pi} \vec{E} \times \vec{B}$$

12. Electric, magnetic fields for dipole radiation:

$$\vec{B} = -\frac{1}{4\pi} \frac{e^{ikr}}{r} \hat{n} \times \frac{d^2 \vec{p}}{dt^2}$$

$$\vec{E} = \vec{B} \times \vec{n}$$