Physics 214. Electricity and Magnetism. Professor Dine

Winter, 2011. Homework Set 1. **Solutions**.

SOLUTIONS

1. Work out the form of the transverse fields for TE case. Discuss the boundary conditions.

   Solution: This is basically described in your text and is a small modification of the lecture material. Start with Jackson's equations 8.23, 8.24. For TE, we have $E_z = 0$. So

   $$\frac{\partial \vec{E}_t}{\partial z} + i\omega \hat{z} \times \vec{B}_t = 0 \qquad \frac{\partial \vec{B}_t}{\partial z} - i\epsilon\omega \hat{z} \times \vec{B}_t = \vec{\nabla}_t B_z. \tag{1}$$

   From the first of these equations, noting the e^{ikz} z-dependence, we have, taking the cross product with $\hat{z}$ and using the triple product identity:

   $$ik\hat{z} \times \vec{E}_t - i\omega \vec{B}_t = 0. \tag{2}$$

   or

   $$\hat{z} \times \vec{E}_t = \frac{\omega}{k}\vec{B}_t. \tag{3}$$

   So, substituting in the second of the two equations in **??**, gives

   $$ik\vec{B}_t - i\epsilon\frac{\omega^2}{k}\vec{B}_t = \vec{\nabla}_t B_z \tag{4}$$

   or

   $$\vec{B}_t = k\frac{\vec{\nabla}_t B_z}{\epsilon\omega^2 - k^2}. \tag{5}$$

   From this, $\vec{E}_t$ is easily constructed using the first of equations **??**.

2. Jackson 8.1

   Solution:

   a) We simply compute the force on the current near the surface of the conductor due to the fields. The force per unit area is:

   $$\vec{P} = \int dz \vec{B} \times \vec{J} = \frac{1}{\mu_c}\int dz \vec{H} \times \vec{J} \tag{6}$$

   $$= \mu_c \int dz \mathrm{Re}(\vec{H}_0 e^{-z(1-i)/\delta + -i\omega t})\frac{1}{\delta}\mathrm{Re}(1-i)(\hat{n} \times \vec{H}_0)e^{-z(1-i)/\delta - i\omega t}).$$

   Here we have used the expressions we derived for the current; we have also dropped terms in the exponents involving kz, since $k \ll \delta$. We need to integrate and time average. Rather than guess, we can simply use Re $z = \frac{z+z^*}{2}$. Then, when we do the integrals over z on the separate terms, the $1-i$ factors cancel out, and we just obtain a factor of $\delta/2$. In more detail, note that when we average over time, we only care about terms where the $e^{-i\omega t}$ cancels against the $e^{i\omega t}$ terms. Correspondingly, the $-iz/\delta$ terms cancel out. The factor $1-i$ then just leaves a 1, after taking the real part. The integral over $e^{-2z/\delta}$ gives a factor of $\delta/2$, and the remaining

1/2 comes from the factor of 1/4 (from taking the real parts) times the two from the two terms where the $e^{-i\omega t}$ factors cancel out.

b) Now consider a plane wave at normal incidence. Because the wave is transverse, both $\vec{E}$ and $\vec{B}$ are parallel to the plane. By the boundary conditions for $\vec{E}$, the tangential components vanish. So if we study the stress tensor (eqn. 6.120) we have that the pressure is

$$T_{zz} = \frac{1}{2}\mu\vec{H}^2 \tag{7}$$

which becomes the equation above, after time averaging.

c) Here, we just note that the time averaged field-squared is 1/2 the peak field-squared.

3. Jackson 8.2

Solution:

a) $\vec{\nabla}_t \times \vec{E} = 0$ (since $E_z = B_z = 0$), and $\vec{\nabla}_t \cdot \vec{E} = 0$. So write

$$E = -\vec{\nabla}_t\phi$$

and

$$= -\nabla_t^2\phi = 0$$

For the two dimensional geometry, this is a familiar elementary physics problem; the electric field points radially,

$$\vec{E} = -A\frac{\hat{r}}{r}$$

and

$$\phi = A\ln(r)$$

Now

$$\vec{B} = \sqrt{\mu\epsilon}\hat{z} \times \vec{E} = A\sqrt{\mu\epsilon}\frac{\hat{z}\times\hat{r}}{r}.$$

So to obtain the energy flow:

$$\hat{z}\cdot\vec{S} = \frac{\mu}{2}A^2\sqrt{\mu\epsilon}\hat{z}\cdot\left(\frac{(\hat{z}\times\hat{r})\times\hat{r}}{r^2}\right) \tag{8}$$

$$= \frac{A^2\mu^{3/2}\epsilon^{1/2}}{2r^2}.$$

Now to get the total flow, we need to integrate over the cross section of the guide. This is trivial, noting $\int d^2a = \int_0^{2\pi} d\phi \int_a^b drr$. So

$$P = \frac{A^2\mu^{3/2}\epsilon^{1/2}}{2}(2\pi)\ln(b/a). \tag{9}$$

We determine the constant A from

$$\vec{H}(a) = A\mu^{1/2}\epsilon^{1/2}\frac{\hat{z}\times\hat{r}}{a} = \vec{H}_0. \tag{10}$$

So

$$A = \frac{H_0 a}{\sqrt{\mu\epsilon}}$$

and

$$P = |H_0|^2a^2\sqrt{\frac{\mu}{\epsilon}}\pi\ln(b/a). \tag{11}$$

b) To obtain the attenuation, we need to determine the loss per unit length, by integrating, again, over the cross sectional *surface* of the guide. This is:

$$\frac{dP}{dz} = \frac{\mu\omega\delta}{4}\left[\int_{r=a} r d\phi |H|^2 + \int_{r=b} r d\phi |H|^2\right] \tag{12}$$

$$= \frac{\mu\omega\delta}{4}\frac{|H_0|^2 a^2}{\mu^2} 2\pi(a^{-1} + b^{-1}).$$

So the attenuation coefficient is:

$$\gamma = \frac{1}{2P}\frac{dP}{dz} = \frac{\mu\omega\delta\epsilon^{1/2}}{4\mu^{1/2}\ln(b/a)}(a^{-1} + b^{-1}). \tag{13}$$

Using $\omega = 2/(\mu\sigma\delta^2)$, we have

$$\gamma = \frac{\epsilon^{1/2}}{2\mu^{1/2}\delta\sigma}\frac{a^{-1} + b^{-1}}{\ln(b/a)}. \tag{14}$$

c) We need the current, say, at b. This is:

$$\hat{n} \times \hat{H} = A\sqrt{\frac{\epsilon}{\mu}}\frac{\hat{r} \times (\hat{z} \times \hat{r})}{b} \tag{15}$$

$$= \frac{A\mu^{-1/2}\epsilon^{1/2}}{b}\hat{z}.$$

But we need to integrate this over the cross section; this gives an extra factor of $2\pi b$, so

$$Z = \frac{\ln(b/a)}{2\pi}\sqrt{\frac{\mu}{\epsilon}}. \tag{16}$$

d) Resistance is similar to the previous problem.

$$\Delta V = IR \tag{17}$$

The change in V per unit length is:

$$\frac{dV}{d\ell} = \phi_0\gamma \tag{18}$$

while the current is $I = 2\pi A\sqrt{\epsilon/\mu}$. $\phi = \ln(b/a)A$, so

$$R = \frac{1}{2\pi\sigma\delta}\left(\frac{1}{a} + \frac{1}{b}\right). \tag{19}$$

4.

Jackson 8.7, parts a and c (you'll need to do part b to do c, but you should be able to make this a bit simpler if you work in the $h \ll a$ limit).

Solution: As in all of our wave guide problems, we have an equation, which follows from the wave equation, of the form:

$$(\nabla^2 + \gamma^2)\psi = 0. \tag{20}$$

However, in this problem, rather than start with $\psi = E_r$ or B_r, it is convenient, as in the text, to study TM modes ($B_r = 0$), and to write an expansion in spherical harmonics (note the spherical symmetry of the problem – the equation separates nicely in spherical coordinates) for B_ϕ. In particular, looking at the equation for the curl in spherical coordinates in the back of your text, we have

$$E_\theta = -\frac{i}{\omega r}\frac{\partial}{\partial r}(rB_\phi) \quad E_r = \frac{i}{\omega r \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta B_\phi). \tag{21}$$

So we readily read off the components of $\vec{E}$ ($E_\phi, B_\theta = 0$) from knowledge of B_ϕ. It is also simple to impose the boundary condition that tangential E (E_θ) vanish. Writing

$$B_\phi = \sum Y_{\ell m}\frac{u_\ell(\omega r)}{r} \tag{22}$$

the requirement $E_\theta = 0$ at the boundary becomes

$$\frac{\partial u_\ell}{\partial r}|_{r=a,b} = 0. \tag{23}$$

It is a simple matter to write the transcendental equation for the frequencies. u_ℓ is a linear combination of r times spherical Bessel functions. To write things compactly, call $A_\ell(r) = rj_\ell$, $B_\ell(r) = rj_\ell(r)$ where j_ℓ and n_ℓ are the spherical Bessel and Neuman functions,

$$u_\ell = \alpha A_\ell + \beta B_\ell \tag{24}$$

then studying the boundary condition at a and b we have

$$A_\ell'(kb)/A_\ell'(ka) = B_\ell'(kb)/B_\ell'(ka) \tag{25}$$

Now consider the case $\ell = 1$. Then we can look up j_1, n_1.

$$j_1 = \frac{\sin x}{x^2} - \frac{\cos x}{x} \qquad n_1 = -\frac{\cos x}{x^2} - \frac{\sin x}{x} \tag{26}$$

We are interested in $a, b \gg h$, and the argument of the Bessel function *large*. So let's write expressions for the derivatives in the limit of large x:

$$A_1'(kr) \approx k(\sin kr + \frac{\cos kr}{kr}) \qquad B_1'(kr) \approx k(-\cos kr + \frac{\sin kr}{kr}) \quad . \tag{27}$$

Here we have dropped terms down by additional powers of $1/r$, i.e. we have only differentiated the cosines and sines in the numerators. At lowest order, then, we have

$$\sin(ka)\cos(kb) - \sin(kb)\cos(ka) = 0 \tag{28}$$

or $\sin k(a-b) = 0$, and $k = n\pi/h$. At next order, write $k = n\pi/h + \Delta k$, where Δk is, by assumption, order $1/a$. Then we have, keeping terms up to the next order in $1/a, 1/b$ (after googling trigonometric identities, **and replacing $1/b$ by $1/a$ in the denominators in our expressions for A_1 and B_1 above)**

$$-\sin k(a-b) - \frac{\cos k(a-b)}{ka} \tag{29}$$

$$\approx -\sin(k(a-b) - \Delta k(a-b)\cos k(a-b) - \frac{\cos k(a-b)}{ka} = 0$$

or

$$\Delta k = \frac{2}{ahk} = \frac{1}{n\pi a}. \tag{30}$$