Physics 214. Electricity and Magnetism. Professor Dine

Winter, 2011. Homework Set 2.

Solutions

1. Study the relativity handout on the course website. Do the exercises

2. Jackson 11.3

   Solution:

   This is easy to show directly. Consider three frames, x, x', x''. Then

   $$x' = (x'' + v_2 t'')\gamma(v_2) \quad t' = (t'' + v_2 x'')\gamma(v_2). \tag{1}$$

   Then

   $$x = (x' + v_1 t')\gamma(v_1) \tag{2}$$

   $$(x'' + v_2 t'' + v_1 v_2 x'')\gamma(v_2)\gamma(v_2)$$

   $$[x''(1 + v_1 v_2) + t''(v_1 + v_2)]\gamma(v_1)\gamma(v_2).$$

   If this is to have the form of a Lorentz transformation with velocity v

   $$\gamma(v) = (1 + v_1 v_2)\gamma(v_1)\gamma(v_2)$$

   $$v\gamma(v) = (v_1 + v_2)\gamma(v_1)\gamma(v_2).$$

   Taking the ratio:

   $$v = \frac{v_1 + v_2}{1 + v_1 v_2}. \tag{3}$$

   It is easy to check that γ^2 satisfies the equations above.

   Note that this also follows from direct multiplication of the matrices γ. You can readily check that with

   $$\Lambda(\omega_1) = \begin{pmatrix} \cosh\omega_1 & \sinh\omega_1 & 0 & 0 \\ \sinh\omega_1 & \cosh\omega_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4}$$

   and similarly for $\Lambda(\omega_2)$. You can check that the product is:

   $$\Lambda(\omega) = \Lambda(\omega_1 + \omega_2) = \begin{pmatrix} \cosh(\omega_1 + \omega_2) & \sinh(\omega_1 + \omega_2) & 0 & 0 \\ (\omega_1 + \omega_2) & (\omega_1 + \omega_2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5}$$

   where you will want to derive identities like

   $$\cosh\omega_1 \cosh\omega_2 + \sinh\omega_1 \sinh\omega_2 = \cosh(\omega_1 + \omega_2).$$

   With a little more fooling around, you can derive the velocity addition law above, but note that the ω's add *very* simply.

3. **Jackson 11.6**

   Solution:

   First note that we would like to study the four velocity of the rocket, and the four acceleration. Invariants such as $(a^\mu)^2$ can be evaluated at any time in any frame. So let's evaluate in the instantaneous rest frame of the rocket. Then

   $$a^1 = \frac{du^1}{d\tau} = g; \quad a^0 = \frac{du^0}{d\tau} \propto \frac{dv}{d\tau} \cdot v = 0$$

   so

   $$a^2 = -g^2$$

   . Now we also have $a \cdot u = 0$ (since $u \cdot u$ is constant). So we can write:

   $$a^0 = \frac{a^1 u^1}{u^0}.$$

   So,

   $$a^2 = (a^0)^2 - (a^1)^2 = (a^1)^2 \left(\frac{(u^x)^2}{1 - (u^x)^2} \right) = \frac{a^1}{(1 - (u^x)^2)} = g^2$$

   $$\frac{u^{x\prime}}{\sqrt{1 - (u^x)^2}} = g \tag{6}$$

   has as solution:

   $$u^x = \sinh(g\tau) \tag{7}$$

   (where we have imposed the boundary condition that the velocity vanish at $\tau = 0$) and leads to the numerical results which most of you found. On each time interval, 83.76 years elapse for the stationary observer. The distance traveled on each interval is 82.79 light years.

4. **Jackson 11.17**

   Solution:

   I liked the idea of this problem: using invariance principles to figure out the rather complicated electric and magnetic fields due to point particles in different situations and frames. Jackson's wording is a bit confusing, and I am not entirely sure how he meant for us to do the problem. In each part, my strategy below is to transform to the rest frame of the charged particle, in which we have simply a static Coulomb field.

   a.
   $$F^{\alpha\beta} = \frac{q}{c} \frac{X^\alpha U^\beta - X^\beta U^\alpha}{[(U_\gamma X^\gamma)^2 - X^\alpha X_\alpha]^{3/2}}. \tag{8}$$

   Now we need simply note that when we transform x'_p, x'_q to the rest frame of the particle, the times will no longer be simultaneous; $x_q = (t_q, \vec{0})$, $x_p = (t_p, \vec{x}_p)$. In the rest frame, $U = (1, \vec{0})$, $U_\alpha X^\alpha = t$, $(U_\alpha X^\alpha)^2 - X_\alpha X^\alpha = \vec{X}^2$. So

   $$F^{0i} = q \frac{-x^i}{(\vec{x}_p^2)^{3/2}} = -E_i \tag{9}$$

   as expected. Similarly $F_{ij} = 0$.

   In the frame in which the particle moves with velocity $v\hat{x}$,

   $$U^\alpha = \gamma(1, v, 0, 0) \quad X^\alpha = (0, vt, b, 0) \tag{10}$$

$$U_\alpha X^\alpha = \gamma(-v^2 t); X^2 = -v^2 t^2 - b^2 \tag{11}$$

so

$$F^{0i} = \frac{q\gamma vt}{(v^2 t^2 \gamma^2 + b^2)^{3/2}}. \tag{12}$$

b. Here we are dealing with the same physical situation, but the space-time points x_p and x_q have different meaning. Now the form in the rest frame is immediately seen to be the Coulomb law, since

$$Y' = (0, \vec{x}_p - \vec{x}_q). \tag{13}$$

Now we want to transform to the frame K. We want to transform by $\vec{v}$. so:

$$x_q = \gamma(t', vt') \quad x_p = (\gamma(t' - v^2 t', \vec{b} - vt') = (\gamma^{-1} t', \vec{b}). \tag{14}$$

Note that the points are not simultaneous. Now consider:

$$Y^\mu = (\gamma^{-1}(1 - \gamma^2)t', \vec{b} - \vec{b}\gamma t'). \tag{15}$$

So we need to evaluate:

$$Y^2 = (\vec{b}^2 + v^2 \gamma^2 t'^2) - \gamma^{-2}(1 - \gamma^2)^2 t'^2 \tag{16}$$

This looks unpromising; the first part looks like what we are after, but there is an extra factor of v^2, and the second looks like a mess. But a miracle happens. With a little algebra:

$$\gamma^{-2} v^2 (1 - \gamma^2)^2 = \frac{v^4}{1 - v^2} \tag{17}$$

and

$$\gamma^2 (v^2 - v^4) t'^2 = v^2 t'^2 = \gamma^2 v^2 t^2. \tag{18}$$

c. Again, it is easiest to think about going to the rest frame. In the moving frame, K', call

$$x_p = (t', \vec{b}) \quad x_q' = (t' - R, \vec{v}(t' - R)). \tag{19}$$

Then, transforming by $-v$, gives:

$$x_q = \gamma^{-1}(t' - R, \vec{0}) \quad \vec{p} = (\gamma g', \vec{b} - \gamma vt'). \tag{20}$$

Actually, we can avoid thinking about the time components all together, by noting that $Z = x_p - x_q$ is timelike, and in the rest frame,

$$(U_\alpha Z^\alpha)^2 = Z^{0\ 2} = \vec{Z}^2 \tag{21}$$

but

$$\vec{Z}^2 = \vec{x}_p^2 \tag{22}$$

so we immediately see that we have the Coulomb expression in the rest frame. We will leave discussing this in the other frame for our discussion in chapter 13.

5. Jackson 11.18. In part c., you need consider only the first of the two vector potentials.

   Solution: We can do this one from scratch. Start with the $\vec{E}$ field in the rest frame,

$$\vec{E} = \frac{q\vec{r}}{r^3}. \tag{23}$$

Now we Lorentz boost this to a frame moving with velocity $v \approx 1$ and large γ. Then, taking the boost along the z axis, E_x, E_y are larger than E_z by a factor γ. Calling the moving frame the primed frame, we can take:

$$z = \gamma(x' - vt') \qquad \vec{x}_\perp = \vec{x}'_\perp \tag{24}$$

$$\vec{E}_\perp = \frac{q\gamma x_\perp}{(x_\perp^2 + \gamma^2(z' - vt')^2)^{3/2}} \tag{25}$$

The fields are largest when $z \approx vt'$. In fact, the expression on the right is, for large γ, a representation of a δ function. To check this, consider the integral:

$$\gamma \int dx \frac{f(x)}{(\gamma^2 x^2 + a^2)^{3/2}} \tag{26}$$

$$= 2\frac{f(0)}{a^2} + \mathcal{O}(1/\gamma).$$

So the result is:

$$\vec{E} = 2q\frac{\vec{r}_\perp}{r_\perp^2}\delta(t' - z'). \tag{27}$$

$\vec{B}$ follows similarly.

Part b is straightforward. For example,

$$\vec{\nabla} \cdot \vec{E} = \rho \approx \vec{\nabla}_\perp \cdot \vec{E}_\perp. \tag{28}$$

The derivatives don't act on the δ function. But $E_\perp$ is just the field of a charge q in two dimensions, multiplied by this delta function. This just corresponds to the time component of the current in your text. The $\vec{\nabla} \times \vec{B}$ equation is similar. Again, only the transverse parts are important. One immediately finds

$$\vec{\nabla} \times \vec{B} = (\vec{\nabla}_\perp \times \vec{B}_\perp) = 2q\vec{v}\vec{\nabla} \cdot (\vec{r}_\perp/r_\perp^2)\delta(t - z) \tag{29}$$

(I have dropped the primes). As above, this is the divergence of the electric field of the two dimensional problem, and the result follows.

For part c,, note, again, that the $\vec{E}$ field is that appropriate to a two dimensional configuration of charge 2. So in evaluating

$$\vec{E} = -\vec{\nabla}A^0 - \frac{\partial \vec{A}}{\partial t} \tag{30}$$

note that the transverse terms give the desired $\vec{E}$ field. The time derivative, acting on the δ function, cancels the z derivative term. A similar analysis gives the $\vec{B}$ field. One can also derive this directly from our Green's function solution, using the form of the current in part b. The z integration gives the δ-function part. The $\vec{x}_\perp$ integration is the integration of the two dimensional static Green's function, which gives a log.