Physics 214. Electricity and Magnetism. Professor Dine

Winter, 2011. Homework Set 5. **Solutions**

1. Determine the cross section for scattering of a linearly polarized wave by a charge carrying out small vibrations under the influence of an elastic force (i.e. an oscillator).

   Solution: This is similar to the case of the free particle we studied in class. Again, we suppose that the wavelength is long, so at the position of the particle we can write:

   $$\vec{E} = \vec{E}_0 e^{-i\omega t} \tag{1}$$

   Then the solution for $\vec{x}$ is:

   $$\vec{x} = \frac{1}{(\omega^2 - \omega_0^2) + i\gamma\omega} e\vec{E}_0 e^{-i\omega t} + \text{transients} \tag{2}$$

   So we can immediately read off the electric dipole moment:

   $$\vec{p} = \frac{e^2 \vec{E}_0}{(\omega^2 - \omega_0^2) + i\gamma\omega} \tag{3}$$

   So we can read off the cross section in the dipole approximation:

2. From the handout on the multipole expansion for radiation, work through the details for the expansion in the case of electric moments, i.e. for $\vec{r} \cdot \vec{E}$. Do this by considering the expansion in the intermediate zone, and matching onto the behavior in the radiation zone. (This is more or less done in the handout, posted on the web, but some details are missing and some formulae may not be reliable at the level of constant factors, etc.)You don't need to be excessively detailed, but try to make clear the connection of the terms in the multipole expansion developed in the intermediate or induction zone and the behavior in the radiation zone.

   Solution: In both cases, the crucial point is that the fields in the intermediate zone are essentially static, so one can use a static multipole expansion. Knowing that the solutions, in general, a spherical Hankel functions, one can then match the short distance and large distance behaviors.

3. Jackson 10.1.

   Solution: In chapter 2.5, Jackson solves the problem of a conducting sphere in a uniform field (appropriate to the dipole approximation). The corresponding field is that of a dipole, aligned with the field, with dipole moment $E_0 a^3$. Similarly, in 10.13, Jackson shows that

   $$\vec{m} = -2\pi a^3 H_{inc} \tag{4}$$

   From this, it follows, as Jackson writes that:

   $$\frac{d\sigma}{d\Omega}(\hat{n}, \hat{\epsilon}; \hat{n}_0, \hat{\epsilon}_0) = k^4 a^6 |\epsilon^* \cdot \epsilon_0 - \frac{1}{2}(\hat{n} \times \epsilon^*) \cdot (\hat{n}_0 \times \hat{\epsilon}_0)|^2 \tag{5}$$

Now we can work out the cross section, summed over final polarizations, by making two explicit choices for ϵ, or by using our trick for the polarization sums. The latter is easier.

Define $\vec{A} = \hat{n}_0 \times \hat{\epsilon}_0$; then we can simplify the expression by taking the ϵ's real in the polarization sums, and dropping the hats over ϵ and n:

$$\frac{d\sigma}{d\Omega} = k^4 a^2 \sum_{pol} |\hat{\epsilon}_i \epsilon_{0i} - \frac{1}{2} \epsilon_i \epsilon_{kli} A_k n_\ell|^2 \tag{6}$$

So using

$$\sum_{pol} \epsilon_i \epsilon_j = \delta_{ij} - n_i n_j \tag{7}$$

we have

$$\frac{d\sigma}{d\Omega} = k^4 a^2 (\delta_{ij} - n_i n_j)[\epsilon_{0i} - \frac{1}{2} \epsilon_{kli} n_\ell A_k][\epsilon_{0j} - \frac{1}{2} \epsilon_{mnj} n_n A_m] \tag{8}$$

Now things simplify in the product. Note that:

$$n_i \epsilon_{kli} = 0; n^2 A^2 = 1; \epsilon_{kli} \epsilon_{0i} n_\ell A_k = n_0 \cdot n \tag{9}$$

and

$$[\hat{n} \times (\hat{n} \times \vec{A})]^2 = [\hat{n} \cdot (\hat{n}_0 \times \hat{\epsilon}_0)]^2 \tag{10}$$

gives Jackson's expression

$$\frac{d\sigma}{d\Omega} = k^4 a^2 [\frac{5}{4} - |\hat{\epsilon}_0 \cdot \hat{n}|^2 - \frac{1}{4} |\hat{n} \cdot (\hat{n}_0 \times \hat{\epsilon}_0)|^2 - \hat{n}_0 \cdot \hat{n}] \tag{11}$$

b. So now for linear polarization, taking $\hat{n}_0 = \hat{z}$, $\hat{\epsilon}_0 = \hat{x}$, we have

$$\hat{\epsilon}_0 \cdot \hat{n} = \sin\theta \cos\phi \qquad \hat{n}_0 \cdot \hat{n} = \cos\theta \quad \hat{n} \cdot (\hat{n}_0 \cdot \hat{\epsilon}_0) = \sin\theta \sin\phi \tag{12}$$

Jackson's result follows with a little algebra.