The Lienard-Wiechart Potentials and the Fields of Moving Charged Particles

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The Lienard-Wiechart Potentials

We can derive the scalar and vector potential for a point charge starting with the expressions we wrote for the scalar and vector potentials,

$$\phi(\vec{x}, t) = \int d^3x' dt' \frac{1}{|\vec{x} - \vec{x}'|} \rho(\vec{x}', t') \delta(t - t' - \frac{1}{c}|\vec{x} - \vec{x}'|). \quad (1)$$

$$\vec{A}(\vec{x}, t) = \int d^3x' dt' \frac{1}{|\vec{x} - \vec{x}'|} \vec{J}(\vec{x}', t') \delta(t - t' - \frac{1}{c}|\vec{x} - \vec{x}'|). \quad (2)$$

and the charge and current distributions we wrote for point charges:

$$\rho(\vec{x}, t) = q\delta(\vec{x} - \vec{x}_o(t)) \qquad \vec{J}(\vec{x}, t) = q\vec{v}_o(t)\delta(\vec{x} - \vec{x}_o(t)) \quad (3)$$

where $\vec{x}_o(t)$ is the position of the particle at time t, and $\vec{v}_o$ is its velocity.

We just need to figure out how to do the integral over the δ-function. For a δ-function, the most we care about is its behavior *near* the point where its argument vanishes. We called t_R the solution to this condition,

$$t_R = t - \frac{1}{c}|\vec{x} - \vec{x}_o(t_R)|. \tag{4}$$

What is somewhat complicated about this equation is that it is an implicit equation for t_R. We can solve it, however, once we know the trajectories of the charged particle. At time $t' = t_R + (t' - t_R)$ near t_R, we can Taylor expand the position:

$$\vec{x}_o(t) \approx \vec{x}_o(t_R) + (t' - t_R)\vec{v}_o(t_R) \tag{5}$$

Using this, we can write:

$$|\vec{x} - \vec{x}_o(t')| \approx |\vec{x} - \vec{x}_o(t_R) - (t' - t_R)\vec{v}_o(t_R)| \tag{6}$$

Call $\vec{\mathcal{R}} = \vec{x} - \vec{x}_o(t_R)$; then

$$|\vec{x} - \vec{x}_o(t')| \approx (\mathcal{R}^2 - 2\vec{\mathcal{R}} \cdot \vec{v}_o(t' - t_R))^{1/2} \tag{7}$$

$$\approx \mathcal{R} - \frac{\vec{\mathcal{R}} \cdot \vec{v}_o}{\mathcal{R}}(t' - t_R)$$

So finally, the argument of the δ-function is:

$$\delta([t - \frac{1}{c}\mathcal{R} - t_R\frac{1}{c}\vec{v}_o \cdot \frac{\vec{\mathcal{R}}}{\mathcal{R}}] - t'(1 - \frac{1}{c}\vec{v}_o \cdot \frac{\vec{\mathcal{R}}}{\mathcal{R}})) \tag{8}$$

Remember that t' is the integration variable and note that t' appears only in the second set of terms. The δ function still vanishes when $t' = t_R$. But what we also need is that:

$$\delta(a(t' - t_R)) = \frac{1}{a}\delta(t' - t_R). \tag{9}$$

So from this we obtain:

$$\phi(\vec{r}, t) = \frac{q}{\mathcal{R} - \frac{1}{c}\vec{v}_o \cdot \vec{\mathcal{R}}} \tag{10}$$

$$\vec{A}(\vec{r}, t) = q\vec{v}\frac{1}{\mathcal{R} - \frac{1}{c}\vec{v}_o \cdot \vec{\mathcal{R}}} \tag{11}$$

where in each case, the quantities on the right hand side are evaluated at the retarded time.

Evaluating the Fields

Our index notation is particularly effective in evaluating the $\vec{E}$ and $\vec{B}$ fields of a point charge. We need to evaluate:

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi \qquad \vec{B} = \vec{\nabla} \times \vec{A}. \tag{12}$$

We need to be careful, however, because t_R is implicitly a function of $\vec{x}$. So when we take derivatives with respect to $\vec{x}$, we need to differentiate not only the terms with explicit $\vec{x}$'s, but also the terms with t_R. So we start by working out these derivatives. Differentiating both sides of:

$$t_R = t - \frac{1}{c}|\vec{x} - \vec{x}_o(t_R)| \tag{13}$$

Remembering that

$$|\vec{x} - \vec{x}_o(t_R)| = ((x_i - x_{oi})^2)^{1/2} \tag{14}$$

gives

$$\partial_i t_R = -\frac{1}{c}\frac{\mathcal{R}_i}{\mathcal{R}} + \frac{\vec{v}_o(t_R) \cdot \vec{\mathcal{R}}}{\mathcal{R}} \partial_i t_R \tag{15}$$

Solving for $\partial_i t_R$:

$$\partial_i t_R = -\frac{\mathcal{R}_i}{c\mathcal{R}} \frac{1}{1 - \frac{\vec{v}_o(t_R)\cdot\vec{\mathcal{R}}}{\mathcal{R}}} \tag{16}$$

It will also be useful to have a formula for $\partial_i \mathcal{R}$. From

$$\mathcal{R} = c(t - t_R) \tag{17}$$

we have

$$\partial_i \mathcal{R} = -c\partial_i t_R. \tag{18}$$

So now we can start taking derivatives.

$$\partial_i \phi = -\frac{c}{(\mathcal{R}c - \vec{\mathcal{R}} \cdot \vec{v})^2} \partial_i (\mathcal{R}c - \vec{\mathcal{R}} \cdot \vec{v}) \tag{19}$$

Now

$$\begin{aligned} \partial_i \vec{\mathcal{R}} \cdot \vec{v} &= \partial_i (r_j - x_{oj}(t_R)) \dot{x}_{oj}(t_R) \\ &= \dot{x}_{oi} - \dot{x}_{oj}^2 \partial_i t_R - \mathcal{R}_j \ddot{x}_{oj} \partial_i t_R \end{aligned} \tag{20}$$

So

$$\partial_i \phi = -\frac{qc}{(\mathcal{R}c - \vec{\mathcal{R}} \cdot \vec{v})^2} (-c \partial_i t_R + v^2 \partial_i t_R + \vec{\mathcal{R}} \cdot \vec{a} \partial_i t_R - v_i) \tag{21}$$

Using our expression for $\partial_i t_R$ gives:

$$\partial_i \phi = \frac{-qc}{(\mathcal{R}c - \vec{\mathcal{R}} \cdot \vec{v})^3} \left[-c^2 \mathcal{R}_i + v^2 \mathcal{R}_i + \vec{\mathcal{R}} \cdot \vec{a} \mathcal{R}_i - v_i (\vec{\mathcal{R}} \cdot \vec{v} - c\mathcal{R}) \right] \tag{22}$$

With a bit more algebra, one can show:

$$\frac{\partial \vec{A}}{\partial t} = \frac{qc}{(\mathcal{R}c - \vec{\mathcal{R}} \cdot \vec{v})^3} \tag{23}$$

$$\left[(\mathcal{R}c - \vec{\mathcal{R}} \cdot \vec{v})(-\vec{v} + \mathcal{R}\vec{a}/c) + \frac{\mathcal{R}}{c}(c^2 - v^2 + \vec{\mathcal{R}} \cdot \vec{a})\vec{v} \right]$$

and combining these, you obtain:

$$\vec{E}(\vec{r}, t) = e \left[\frac{\hat{n} - \vec{v}}{\gamma^2 (1 - \vec{v} \cdot \vec{n})^3 R^2} \right] + e \left[\frac{\hat{n} \times [(\hat{n} - \vec{v}) \times \dot{\vec{v}}]}{(1 - \vec{v} \cdot \vec{n})^3 R} \right] \tag{24}$$

where $\hat{n} = \frac{\vec{R}}{R}$. All quantities on the right hand side are to be evaluated at the retarded time.

Similarly,

$$\vec{B} = \frac{1}{c}\hat{\mathcal{R}} \times \vec{E}(\vec{r}, t). \tag{25}$$

Exercise: Fill in the details of the calculations of $\vec{E}$ and $\vec{B}$, using the index notation as above.

More Covariant Derivation

The Green's function can be written in a covariant-looking fashion:

$$G(x) = 2\delta(x^2)\theta(x^0) \tag{26}$$

To check this, note

$$\delta(x^2) = \delta(t^2 - \vec{x}^2) \tag{27}$$

This has roots at $t = \pm|\vec{x}|$; because of the θ function, we keep only the positive root. Then using the rules for δ-functions of functions,

$$G(x) = \frac{1}{t}\delta(t - |\vec{x}|)\theta(t). \tag{28}$$

So

$$A^\mu(x) = \int d^4x' G(x - x') j^\mu(x') \tag{29}$$

We can also write the current associated with a charge in the covariant manner:

$$j^\mu(x) = \int d\tau u^\mu(\tau) \delta(x - x_0(\tau)) \tag{30}$$

(again, you should check the components if this is not familiar).

So we have

$$A^\mu(x) = 2e \int d^4x' \int d\tau u^\mu(\tau)\theta(x^0 - x_0^0(\tau))\delta((x - x_0(\tau)^2) \quad (31)$$

where we have used the δ-function in j^μ. To do the τ integral, we note:

$$\frac{\partial}{\partial\tau}(x - x_0(\tau))^2 = -2u^\mu(x - x_0)_\mu \quad (32)$$

so

$$A^\mu = e\frac{u^\mu(\tau)}{u \cdot (x - x_0)} \quad (33)$$

which is what we found previously. (Note, due to the δ function, $\mathcal{R} = 0$).

Some Checks of These Results

We should be able to reproduce our earlier results for radiation by a dipole and for the fields of a particle in uniform motion. Consider, first, a non-relativistic particle undergoing acceleration. For $\vec{E}$, we have

$$\vec{E}(\vec{x},t) \approx e\frac{\hat{n}\times(\hat{n}\times\dot{v})}{R} \tag{34}$$

$$= \frac{e}{r}[\hat{n}(\hat{n}\cdot\ddot{\vec{p}}) - \ddot{\vec{p}}].$$

This is our earlier result (similarly for $\vec{B}$; again, everything on the right hand side is evaluated at the retarded time).

Particle in Uniform Motion

As in chapter 12 of Jackson, we consider a particle moving along the x axis with velocity v, and observe its motion at the point $(0, b, 0)$ at time t. The crucial issue is to keep track of the retarded time in our expression for $\vec{E}$. We can do this as in Jackson's 14.2, or we can proceed by actually solving for the retarded time, which is instructive. The retarded position and time can be labeled

$$(t', vt', 0, 0) \tag{35}$$

and satisfies:

$$(t' - t)^2 - (b^2 + v^2 t'^2) = 0; \quad R^\mu = (t - t', vt', b, 0). \tag{36}$$

This is a quadratic equation for t'. The solution is not particularly pretty:

$$t' = \frac{2t - \sqrt{4t^2 + 4(b^2 - t^2)(1 - v^2)}}{2(1 - v^2)} \tag{37}$$

(note that the negative sign root of the quadratic equation is necessary so that $t' < t$).

$$= \gamma^2 t - \sqrt{b^2\gamma^{-2} + v^2 t^2}$$

To determine the electric field, we need $\hat{n} \cdot \vec{R}$:

$$\hat{n} = \frac{(-vt', b, 0)}{\sqrt{b^2 + v^2 t'^2}}; \quad \hat{n} \cdot \vec{v} = \frac{v^2 t'}{\sqrt{b^2 + b^2 t'^2}}. \tag{38}$$

Also, $\hat{n} \cdot \vec{v}\ R = v^2 t'$, so

$$\begin{aligned}(R - \hat{n} \cdot \vec{R} v) = (t - t' - v^2 t') = t - (1 - v^2) t' = \sqrt{b^2 \gamma^{-2} + 4 v^2 t^2} \\ = \gamma^{-1} \sqrt{b^2 + 4 v^2 \gamma^2 t^2}\end{aligned} \tag{39}$$

So, for example,

$$E_y = \frac{e \gamma b}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \tag{40}$$

as expected.

Energy Emission by an Accelerating Particle

In the frame where the particle is at rest at a given instant, the energy emitted in a time interval *dt* is just what we found earlier (and is given again by our formulas above)

$$d\mathcal{E} = \frac{2e^2}{3}\dot{v}^2 dt \tag{41}$$

The total momentum radiated in a similar time interval is zero. This is easily seen by considering the stress tensor. The momentum flux in the *i*'th direction.

$$T_{ij} = E_i E_j + B_i B_j - \frac{1}{2}\delta_{ij}(\vec{E}^2 + \vec{B}^2). \tag{42}$$

So

$$n_i T_{ij} = -\frac{1}{2} n_j (\vec{E}^2 + \vec{B}^2). \tag{43}$$

(We have used the transversality of the radiation field). Integrated over angles, this gives zero ($\vec{E} \to \vec{E}, \vec{B} \to -\vec{B}$) under $\vec{n} \to -\vec{n}$).

So dP^0 is the time component of a four vector, but $\frac{dP^0}{dt}$ is the same in any frame, i.e. it is a scalar. We can see this by writing:

$$dP^\mu = -2\frac{e^2}{3}\frac{du^\nu}{ds}\frac{du_\nu}{ds}dx^\mu = -\frac{2e^2}{3}\frac{du^\nu}{ds}\frac{du_\nu}{ds}dx^\mu \tag{44}$$

(check in rest frame!). We can write

$$\frac{dP^0}{dt} = -2\frac{e^2}{3c^3m^2}\frac{dp^\nu}{ds}\frac{dp^\nu}{ds} \tag{45}$$

(Lorentz invariant, as claimed earlier).

After some algebra,

$$\frac{dp^\nu}{d\tau}\frac{dp^\nu}{d\tau} = m^2\gamma^6[(v \times \dot{v})^2 - \dot{v}^2] \tag{46}$$

and

$$P \equiv \frac{dP^0}{dt} = \frac{2}{3}\frac{e^2}{c}\gamma^6[(v \times \dot{v})^2 - \dot{v}^2]. \tag{47}$$

Comparison of Linear, Circular Acceleration

From these formulas, one can see that, for a given applied force, particles in circular motion radiate much more rapidly than particles in linear motion. Again writing

$$P = -\frac{2}{3}\frac{e^2}{m^2}\frac{dp^\mu}{d\tau}\frac{dp_\mu}{d\tau} \tag{48}$$

we have, for linear motion,

$$P = \frac{2}{3}\frac{e^2}{m^2}\left(\frac{d\vec{p}}{d\tau}\right)^2 - \left(\frac{dE}{d\tau}\right)^2 \tag{49}$$

but $\frac{dE}{d\tau} = \frac{d}{d\tau}\gamma = \dot{v}v\gamma^3$, while $\frac{dp}{d\tau} = \dot{v}\gamma^3$ so

$$P = -\frac{2}{3}\frac{e^2}{m^2c^3}\left[\left(\frac{d\vec{p}}{d\tau}\right)^2 - \beta^2\left(\frac{dp}{d\tau}\right)^2\right] \tag{50}$$

So we can compare linear, circular acceleration.

Linear:

$\frac{dE}{dx} \approx \frac{dp}{dt}$, so

$$P = \frac{2}{3}\frac{e^2}{m^2 c^3}\left(\frac{dp}{dt}\right)^2 . \tag{51}$$

Circular:

$|\frac{d\vec{p}}{d\tau}| = \gamma\omega|\vec{p}|$, so

$$P = \frac{2}{3}\frac{e^2 c}{\rho^2}\beta^4\gamma^4 \tag{52}$$

Reason why circular accelerators for electrons problematic.

Now we look at the radiation in more detail, particularly the angular and frequency distribution. To do this we return to the expression for the $\vec{E}$ field (radiation part):

$$\vec{E}(\vec{r},t) = e\left[\frac{\hat{n}\times[(\hat{n}-\vec{v})\times\dot{\vec{v}}]}{(1-\vec{v}\cdot\vec{n})^3 R}\right] \tag{53}$$

and recall $\vec{S} = \hat{n}|\vec{E}|^2$, so

$$\vec{S}\cdot\hat{n} = \frac{e^2}{4\pi c}\frac{1}{R^2}\left|\left[\frac{\hat{n}\times[(\hat{n}-\vec{v})\times\dot{\vec{v}}]}{(1-\vec{v}\cdot\vec{n})^3 R}\right]\right|^2 \tag{54}$$

Note that in n.r. limit, denominator is one, but in ultrarelativistic limit, the denominator vanishes in the forward direction – the radiation is highly peaked.

It is convenient to change variables so that integrals over time (e.g. for the total energy) are written in terms of the retarded time.

$$\int dt = \int \frac{\partial t}{\partial t'} dt' \tag{55}$$

Recall

$$\frac{\partial t'}{\partial t} = (1 - \hat{n} \cdot \vec{v})^{-1}. \tag{56}$$

(from $R = |\vec{x} - \vec{x}_0(t')$, $\frac{\frac{\partial R}{\partial t'} = -\vec{v} \cdot \vec{R}}{R}$, so, since $t' = t - R$,

$$\frac{\partial t'}{\partial t} = 1 - \frac{\partial R}{\partial t'} \frac{\partial t'}{\partial t}, \tag{57}$$

giving the result above. So, in particular,

$$\frac{dP(t')}{d\omega} = R^2 \vec{S} \cdot \hat{n}(1 - \vec{v} \cdot \hat{n}). \tag{58}$$

Motion in a straight line (linear accelerator)

Here $\vec{v} \times \frac{d\vec{v}}{dt} = 0$.

$$\frac{dP(t')}{d\omega} = \frac{e^2}{4\pi}\dot{v}^2 \frac{\sin^2\theta}{(1 - v\cos\theta)^5}. \tag{59}$$

For $v \to 1$, the denominator is

$$1 - v + \frac{1}{2}\theta^2 \approx \frac{1}{2}(\gamma^{-2} + \theta^2) \approx \frac{\gamma^{-2}}{2}(1 + \gamma^2\theta^2)$$

again indicating the strong forward peaking; $\theta \sim \gamma^{-1}$.
Integrating over angles,

$$P(t') = \gamma^6 \frac{2}{3}\frac{e^2\dot{v}^2}{c^3} \tag{60}$$

The integrals above are elementary. For example

$$\int d\theta \frac{\theta^3}{2} \frac{\dot{v}^2}{(1+\gamma^2\theta^2)^5} \tag{61}$$

$$= \gamma^6 \frac{\dot{v}^2}{2} \int_0^\infty dx \frac{x^3}{(1+x^2)^5} = \frac{\dot{v}^2}{24}.$$

Circular motion

Highly peaked in angle as before. Take $\vec{b} \perp \dot{\vec{v}}$. Evaluate:

$$\frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c} \frac{|\hat{n} \times [(\hat{n} - \vec{v}) \times \dot{v}]|^2}{(1 - \hat{n} \cdot \vec{v})^5}. \tag{62}$$

The numerator may be written, using our identity for ϵ's (in particular, $|\vec{A} \times \vec{B}|^2 = |\vec{A}|^2 |\vec{B}|^2 - (\vec{A} \cdot \vec{B})^2$):

$$|\hat{n} \times [(\hat{n} - \vec{v}) \times \dot{v}]|^2 = |(\hat{n} - \vec{v}) \times \dot{\vec{v}}|^2 - |\hat{n} \cdot [(\hat{n} - \vec{v}) \times \dot{\vec{v}}]|^2 \tag{63}$$

$$= (1 + v^2 - 2\hat{n} \cdot \vec{v})\dot{\vec{v}} - (\hat{n} \cdot \dot{\vec{v}})^2 - |\hat{n} \cdot (\vec{v} \times \dot{\vec{v}})|^2.$$

Taking $\vec{v}$ along the z axis, and $\dot{\vec{v}}$ along the y axis, and $\hat{n} = (\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi))$,

$$\frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c^3} \frac{|\dot{\vec{v}}|^2}{(1 - v\cos\theta)^3} [1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1 - v\cos(\theta))^2}]. \tag{64}$$

Again, highly peaked in forward direction. Integrating over angles as before,

$$P(t') = \frac{2}{3}\frac{e^2|\dot{\vec{v}}|^2}{c^3}\gamma^4. \tag{65}$$

To compare with linear acceleration, we write in terms of

$$\frac{d\vec{p}}{dt} = \gamma m\dot{\vec{v}} \tag{66}$$

so

$$P(t') = \frac{2}{3}m^2\frac{e^2}{c^3}\gamma^2\dot{\vec{p}}^2. \tag{67}$$

This is larger than the linear case by γ^2 for the same force.

Distribution in Frequency (and angle)

I'll follow Jackson and define (not to be confused with the vector potential)

$$\vec{A} = \left(\frac{c}{4\pi}\right)^{1/2} R\vec{E} \tag{68}$$

so

$$\frac{dP}{d\Omega} = |A(t)|^2. \tag{69}$$

Fourier transforming, following Jackson's convention:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} f(\omega) \tag{70}$$

we have, for the *total* energy radiated during the course of the particle motion:

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} |\vec{A}(\omega)|^2 d\omega \tag{71}$$

$$\equiv \int_0^{\infty} \frac{d^2 I(\omega, \hat{n})}{d\omega d\Omega} d\omega$$

where, since $\vec{A}(\omega) = \vec{A}^*(-\omega)$

$$\frac{d^2 I}{\cdots} = 2|\vec{A}(\omega)|^2 \tag{72}$$

We can calculate $\vec{A}$ directly given knowledge of the trajectory.

$$\vec{A}(\omega) = \left(\frac{e^2}{8\pi^2 c}\right) \int_{-\infty}^{\infty} e^{i\omega t} \left[\frac{\hat{n} \times (\hat{n} - \vec{\beta})\dot{\beta}}{(1 - \vec{\beta} \cdot \hat{n})^2}\right] dt \qquad (73)$$

(I have given in and used Jackson's $\vec{\beta}$ notation). Note that the quantities on the right hand side are to be evaluated at the retarded time, but here we can just change variables in the integral.

$$\frac{dt'}{dt} = (1 - \vec{\beta} \cdot \hat{n})^{-1}. \qquad (74)$$

When we do this, we replace $e^{i\omega t}$ by $e^{i\omega t' - R(t')/c} \approx e^{i\omega t - \hat{n} \cdot \vec{r}(t')}$ ($\vec{r}$ is the position of the particle).

So we are left with

$$\vec{A}(\omega) = \left(\frac{e^2}{8\pi^2 c}\right)^{1/2} \int_{-\infty}^{\infty} dt e^{i\omega(t-\hat{n}\cdot r(t)/c)} \left[\frac{\hat{n}\times(\hat{n}-\vec{\beta})\dot{\beta}}{(1-\vec{\beta}\cdot\hat{n})^2}\right] \quad (75)$$

Amazingly the term in braces is a total derivative

$$\frac{d}{dt}\frac{\hat{n}\times(\hat{n}\times\vec{\beta})\times\dot{\vec{\beta}}}{(1-\vec{\beta}\cdot\hat{n})} = \left[\frac{\hat{n}\times(\hat{n}-\vec{\beta})\dot{\beta}}{(1-\vec{\beta}\cdot\hat{n})^2}\right],$$

and after an integration by parts one obtains:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty}\hat{n}\times(\hat{n}\times\vec{\beta})e^{i\omega(t-\hat{n}\cdot\vec{r}(t)/c)}dt\right|^2. \quad (76)$$

As long as there is some component of $\dot{\vec{v}}$ not parallel to $\vec{v}$, this dominates and at any instant, the motion can be treated as circular. One then obtains expressions for the radiation by a "straightforward" integration.

Brief aside: It is interesting that eqn. 75 doesn't involve the acceleration. But it better vanish if $\dot{\vec{\beta}}$ is zero. This is easy to see; the integrand, if $\vec{\beta}$ is a constant, is

$$\frac{d}{dt}\left[\frac{\hat{n}\times(\hat{n}\times\vec{\beta})}{1-\hat{n}\cdot\vec{\beta}}\right]e^{i\omega(t-\hat{n}\cdot\vec{r})} \tag{77}$$

(here, and above, it is important that $\dot{\vec{r}} = \vec{\beta}$). The integral thus vanishes.

To set up the problem, it is helpful to set up coordinates, e.g. as in Jackson, and to work out explicitly the various quantities appearing here. Taking

$$\vec{r} = \hat{x}\sin(\omega t/\rho) + \hat{y}\cos(vt/\rho) \tag{78}$$

and taking the vector $\hat{n}$ in the (x, z) plane

$$\hat{n} = \hat{x}\cos\theta + \hat{y}\sin\theta \tag{79}$$

(I am following Jackson in making a slightly unconventional choice for θ, but it makes $\theta \to 0$ the region where most of the radiation lies). It is also helpful to define two polarization vectors:

$$\hat{\epsilon}_{\|} = \hat{y}; \hat{\epsilon}_{\perp} = \hat{z}\sin\theta - \hat{x}\cos\theta \tag{80}$$

one can work out all of the quantities appearing in the integrand. The results can be expressed as modified Bessel functions.