# Physics 214. Electricity and Magnetism Professor Dine

## Winter 2011: MIDTERM EXAM

**Do Three of four problems: fourth will be graded for Extra Credit**

**Problem 1**

a) For a waveguide with a square cross section (each side of length $a$), with dielectric constant $\epsilon$ and magnetic permeability 1, write down the lowest TM mode. For this mode, determine the dispersion relation and cutoff frequency.

**Solution:** For TM modes, the boundary condition is simple: $E_z = \psi = 0$. One can just start with the wave equation

$$\left[\frac{\partial^2}{\partial z^2} + \vec{\nabla}_t^2 - \frac{\epsilon}{c^2}\frac{\partial^2}{\partial t^2}\right]\psi(\vec{x}_t)e^{ikz-i\omega t} = 0. \tag{1}$$

so

$$\left[\frac{\epsilon\omega^2}{c^2} - k^2 + \vec{\nabla}_t^2\right]\psi(x,y) = 0. \tag{2}$$

We can write down the solutions simply (if this is not clear, you should go through the exercise of separating variables):

$$\psi = A_{mn}\sin(\frac{n\pi x}{a})\sin(\frac{m\pi y}{a}). \tag{3}$$

(compare this with the problem of the infinite square well in two dimensions in quantum mechanics, with which it is mathematically identical), with

$$\frac{\epsilon\omega^2}{c^2} = k^2 + \frac{\pi^2}{a^2}(m^2+n^2). \tag{4}$$

This is the dispersion relation. One can use it to calculate the phase and group velocities. The requirement that $k^2$ is positive means:

$$\omega^2 > \omega_c^2 = \frac{c^2}{\epsilon}\frac{\pi^2}{a^2}(m^2+n^2). \tag{5}$$

b) Evaluate the transverse components of the electric and magnetic fields for this mode, using

$$\vec{E}_t = \frac{i}{\epsilon\omega^2 - k^2}\left[k\vec{\nabla}_t E_z - \omega\hat{z}\times\vec{\nabla}_t B_z\right] \tag{6}$$

and

$$\vec{B}_t = \frac{i}{\epsilon\omega^2 - k^2}\left[k\vec{\nabla}_t B_z + \epsilon\omega\hat{z}\times\vec{\nabla}_t E_z\right] \tag{7}$$

Verify that $\vec{\nabla}\cdot\vec{E} = 0$. Calculate the Poynting vector (component along the guide).

**Solution:** This is a pretty simple plug in. E.g.

$$\vec{E}_t = \frac{ikA_{mn}}{\epsilon\omega^2 - k^2}\left(\hat{x}\frac{m\pi}{a}\cos(\frac{m\pi x}{a})\sin(\frac{n\pi y}{a}) + \hat{y}\frac{n\pi}{a}\sin(\frac{m\pi x}{a})\cos(\frac{n\pi y}{a})\right). \tag{8}$$

$$\vec{B}_t = \frac{i\epsilon\omega A_{mn}}{\epsilon\omega^2 - k^2}\left(\hat{y}\frac{\pi x}{a}\cos(\frac{m\pi x}{a})\sin(\frac{n\pi y}{a}) - \hat{x}\frac{n\pi}{a}\cos(\frac{n\pi y}{a})\cos(\frac{m\pi x}{a})\right) \tag{9}$$

It is straightforward to check $\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\vec{B} = 0$ and to construct the Poynting vector. E.g.

$$\vec{\nabla}\cdot\vec{E} = ik\psi - \frac{ikA_{mn}}{\epsilon\omega^2 - k^2}\left((\frac{m\pi}{a})^2\sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{a}) + (\frac{n\pi}{a})^2\sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{a})\right). \tag{10}$$

This vanishes, due to the dispersion relation.

$$\vec{\nabla}\cdot\vec{B} \propto \cos(\frac{m\pi x}{a})\cos(\frac{n\pi y}{a}) - \cos(\frac{m\pi x}{a})\cos(\frac{n\pi y}{a}) = 0. \tag{11}$$

c) If the skin depth is $\delta$ and the conductivity $\sigma$, estimate, without deriving or quoting complicated formulas, the attenuation length of signals in the guide. Remember the basic formula that the power loss per unit area is

$$\frac{dP}{da} = \frac{\omega\delta}{4}|B_{\|}|^2. \tag{12}$$

The result should depend just on $\sigma$, $\delta$, and $a$.

**Solution:** A very crude estimate for $\frac{dP}{da}$ assumes that $k^2 \sim \epsilon\omega^2 \sim \gamma^2$. Then $B \sim \psi \sim E_t$, so

$$\oint \frac{dP}{da} \approx a\omega\delta\psi^2. \tag{13}$$

On the other hand, the energy flux behaves as $\epsilon\psi^2 a^2$. So

$$\frac{1}{\beta} = \frac{\oint \frac{dP}{da}}{\int d^2aP} \sim \frac{\epsilon\delta\omega}{a} \tag{14}$$

Note, in particular, $\beta \propto 1/\delta$, i.e. it gets longer with $\delta$; at the same time it grows with $a$ (reflecting the ration of the perimeter to the area).

**Problem 2.**
a. Work out the components of the tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ in terms of those of $\vec{E}$ and $\vec{B}$.

**Solution:**

$$F_{0i} = \partial_0 A_i - \partial_i A^0 = -\partial_0 A^i - \partial_i A^0 = E_i$$
$$F_{ij} = -\partial_i A^j - \partial_j A^i = -\epsilon_{ijk}B^k.$$

b. Work out the components of

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}. \tag{15}$$

**Solution:**

$$\begin{aligned}\tilde{F}_{0i} &= \frac{1}{2}\epsilon_{0ijk}F^{jk}\\ &= -\frac{1}{2}\epsilon_{0ijk}\epsilon_{jk\ell}B^\ell\\ &= -\frac{1}{2}(\delta_{i\ell}\delta_{kk} - \delta_{ik}\delta_{\ell k})B^\ell\\ &= -B^i\end{aligned}$$

. Similarly,

$$\tilde{F}_{ij} = \frac{1}{2}\times 2\epsilon_{ij0\ell}F^{0\ell}$$

$$= \epsilon_{0ij\ell} F^{0\ell}$$
$$= -\epsilon_{ij\ell} E^{\ell}.$$

c. Imagine you were told that Maxwell's equations were:

$$\partial_\mu F^{\mu\nu} = 0 \qquad \partial_\mu \tilde{F}^{\mu\nu} = 4\pi J^\nu_{mag}. \tag{16}$$

Write the equations in terms of $\vec{E}$ and $\vec{B}$, and the charge density and three-vector current. Interpret. Explain why this is just a renaming of the fields and charges/currents we know.

**Solution:** Using our result in part (b), these equations read:

$$-\vec{\nabla} \cdot \vec{B} = 4\pi\rho_{mag} \qquad \vec{\nabla} \times \vec{E} = 4\pi \vec{J}_{mag} - \frac{\partial \vec{B}}{\partial t}.$$

These become Maxwell's equations if we make the replacement:

$$\vec{E} \to \vec{B}; \vec{B} \to -\vec{E}; \rho_{mag} \to \rho; \vec{J}_{mag} \to \vec{J}$$

This is slightly different then Jackson, whose equations correspond to

$$\partial_\mu F\tilde{F}^{\mu\nu} = -4\pi J^\nu_{mag}.$$

As a result, his duality transformation is slightly different. So if there are *only* magnetic charges, electrodynamics is equivalent to what we usually write.

**Problem 3.** An electron of energy 200 GeV in the lab collides with a positron of energy $x$ GeV producing a $Z$ meson, which has a mass of 91 GeV. What is $x$? What is the velocity of the $Z$ meson? (Viewed in the lab frame, i.e. the frame in which the electron and positron have the specified energies?). You can neglect the mass of the electron in this problem (what does this say about the energy and momentum – use units with $c = 1$!!) Remember that in the rest frame of the $Z$ meson, the total energy is 91 GeV.
bf Solution:
The total energy in the center of mass must be $M_Z^2$. This is most easily evaluated using invariants. In particular, we know that

$$s = (p_e + p_{\bar{e}})^2 \tag{17}$$

is the center of mass energy squared, as can be seen by going to the rest frame of the *system*. $p_e, p_{\bar{e}}$ are the four momenta of the electron and positron. Neglecting the mass of the electron,

$$s = 2p_e \cdot p_{\bar{e}} = 2(E_e E_{\bar{e}} - \vec{p}_e \cdot \vec{p}_{\bar{e}}) = 4|\vec{p}_e||p_{\bar{e}}| \tag{18}$$

where we have noted that the electron and positron move in opposite directions in the lab frame, and the momenta and energy are approximately equal. So

$$m_Z^2 = 4x \times 200 \text{ GeV}. \tag{19}$$

So

$$x = 10.4 \text{GeV}. \tag{20}$$

The velocity of the center of mass in the lab frame is the total momentum divided by the energy, or

$$v = 189.6/210.4 \approx 0.9. \tag{21}$$

**Problem 4.**

We studied the motion of a relativistic charged particle in perpendicular electric and magnetic fields, using the trick of transforming to a particular frame. Consider the problem of a *non-relativistic* particle in such fields, but instead of being too clever, just write the equations of motion and solve. You may want to simply guess the form of a solution (remembering what we found in the relativistic case – uniform drift along the $\vec{E} \times \vec{B}$ direction), and simply plug in and determine the form of the various coefficients in your guess.

**Solution:** This is just an exercise with the Lorentz force law. Taking $\vec{E} = E\hat{x}$; $\vec{B} = B\hat{y}$, we have

$$m\frac{dv_z}{dt} = qv_x B; \quad m\frac{dv_x}{dt} = qE - qv_z B; \quad \frac{dv_y}{dt} = 0. \tag{22}$$

It is easy to see (e.g. by decoupling the second and third equations by taking an additional time derivative, or just plugging in a guess motivated by our discussion in class) that

$$v_y = \text{constant}; \; v_x = a\cos(\omega t); \quad v_z = a\sin(\omega t) + \frac{E}{B}. \tag{23}$$

Here $\omega$ is the cyclotron frequency, $\omega^2 = \frac{q^2 B^2}{m^2}$. So indeed we have constant drift in the direction $\vec{E} \times \vec{B}$, drift along the direction of $E$, and circular motion about the direction of the $B$ field.